Ćwiczenia nr 12

środa, 26 maja 2021

R-nia różniczkowe liniowe I rzędu

$$y' + p(x) \cdot y = q(x)$$

Np. R-nia liniowe:

- $y' - 2xy = x^3$
  $p(x) = -2x, \quad q(x) = x^3$
- $y' + \frac{y}{x} = xe^x$
  $p(x) = \frac{1}{x}, \quad q(x) = xe^x$
- $y' - x = e^x y$
  $y' - e^x y = x$
  $p(x) = -e^x, \quad q = x$

(1) $y' = 2x(y-1)$

RLN: (*) $y' - 2xy = -2x$

$p(x) = -2x$, $q(x) = -2x$

r-nie liniowe niejednorodne ($q(x) \neq 0$)

jednorodne: $q(x) = 0$

RLJ: $y' + p(x)y = 0$

Schemat rozwiązania:

1° równanie jednorodne

RJ: $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{y} = 2x\,dx$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = x^2 + C$$

$$|y| = e^{x^2 + C}$$

Wzór:

$$\boxed{y = Ce^{-\int p(x)\,dx}}$$

CORJ

$$|y| = e^{x^2+C}$$

$$\boxed{y = Ce^{x^2}}$$ CORJ (RORJ)

całka ogólna równania jednorodnego

2° metoda uzmienniania stałej

Niech $C = C(x)$

wtedy $y = C(x) \cdot e^{x^2}$ (**)

oraz $y' = C'(x) \cdot e^{x^2} + 2x\,C(x) \cdot e^{x^2}$

Wtedy r-nie (*) ma postać:

$$\underbrace{C'(x)e^{x^2} + \cancel{2xC(x)e^{x^2}}}_{y'} - 2x\,\underbrace{\cancel{C(x)\cdot e^{x^2}}}_{y} = -2x$$

Uwaga! wyrażenia z $C(x)$ upraszczają się!

$$C'(x)e^{x^2} = -2x \Rightarrow C'(x) = -2xe^{-x^2} \Rightarrow C(x) = \int -2xe^{-x^2}dx = e^{-x^2} + C_1$$

$$C(x) = e^{-x^2} + C_1$$

Tak więc (**) $y = (e^{-x^2} + C_1)e^{x^2}$

$$y = e^{-x^2} \cdot e^{x^2} + C_1 e^{x^2}$$

$$\boxed{y = 1 + Ce^{x^2}}$$

CORN

całka ogólna równania niejednorodnego

② $y' + y\,\mathrm{ctg}x = \sin x$

$p(x) = \mathrm{ctg}(x)$, $q(x) = \sin x$

$y' + y\,\mathrm{ctg}x = \sin x$ $\qquad p(x) = \mathrm{ctg}(x), \quad q(x) = \sin x$

1° RJ: $\quad y' + y\,\mathrm{ctg}x = 0$

$$\frac{dy}{dx} = -y\,\mathrm{ctg}x \quad / \cdot \frac{dx}{y}, \quad y \neq 0$$

$$\int \frac{dy}{y} = -\int \mathrm{ctg}x\,dx$$

$$\ln|y| = -\int \frac{\cos x}{\sin x}\,dx \qquad \boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

$$\ln|y| = -\ln|\sin x| + C$$

$$\ln|y| = \underbrace{C}_{\ln C_1,\ C_1 > 0} - \ln|\sin x|$$

$$\ln|y| = \ln C_1 - \ln|\sin x| \qquad (\ln C_1 + \ln|\sin x|^{-1})$$

$$\ln C_1 \cdot |\sin|)^{-1} = \ln \frac{C_1}{|\sin x|}$$

$$\ln|y| = \ln \frac{C_1}{|\sin x|}$$

$$|y| = \frac{C_1}{|\sin x|}$$

$$y = \frac{C_2}{\sin x}$$

ostatecznie $\underline{y = \dfrac{C}{\sin x}}$

CORJ

2° $\quad C = C(x)$

$$y = \frac{C(x)}{\sin x} \Rightarrow y' = \left(\frac{C(x)}{\sin x}\right)' = \frac{C'(x)\sin x - C(x)\cos x}{\sin^2 x} = \frac{C'(x)}{\sin x} - \frac{C(x)\cos x}{\sin^2 x}$$

$$y = \frac{C(x)}{\sin x} \Rightarrow y' = \left(\frac{C(x)}{\sin x}\right)' = \frac{C'(x)\sin x - C(x)\cos x}{\sin^2 x} = \frac{C'(x)}{\sin x} - \frac{C(x)\cos x}{\sin^2 x}$$

R-nie:
$$\underbrace{\frac{C'(x)}{\sin x} - \cancel{\frac{C(x)\cos x}{\sin^2 x}}}_{y'} + \cancel{\underbrace{\frac{C(x)}{\sin x}}_{y} \cdot \frac{\cos x}{\sin x}} = \sin x$$

$$\frac{C'(x)}{\sin x} = \sin x \Leftrightarrow C'(x) = \sin^2 x$$

$$C(x) = \int \sin^2 x \, dx$$

$$C(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Ostatecznie:
$$y = \frac{\frac{1}{2}x - \frac{1}{4}\sin 2x + C}{\sin x}$$ CORN

(3)
$$y' - 2y = xe^{3x}$$

$p(x) = -2$ – stała $\qquad q(x) = x \cdot e^{3x}$

1° RJ: $y' - 2y = 0$

$$\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2dx$$

$$\int \frac{dy}{y} = 2\int dx \Rightarrow \ln|y| = 2x + C$$

$$\int \frac{dy}{y} = 2\int dx \Rightarrow \ln|y| = 2x + C$$

$$|y| = e^{2x+C}$$

$$\underbrace{y = Ce^{2x}}_{\text{CORJ}}$$

2° metoda przewidywań

Przewiduję $y = (ax+b)\cdot e^{3x}$, $a,b \in \mathbb{R}$ $a,b = ?$

$$y' = ae^{3x} + 3(ax+b)e^{3x}$$

$$\underbrace{ae^{3x} + 3axe^{3x} + 3be^{3x}}_{y'} - 2\underbrace{(ax+b)e^{3x}}_{y} = xe^{3x}$$

$$ae^{3x} + 3axe^{3x} + 3be^{3x} - 2axe^{3x} - 2be^{3x} = xe^{3x} \quad /: e^{3x}$$

$$a + 3ax + 3b - 2ax - 2b = x$$

$$ax + a + b = x$$

x: $a = 1$

w: $a + b = 0 \Rightarrow b = -1$

Rozwiązanie (całka) szczególne(-a) RN: $\underbrace{y = (x-1)e^{3x}}_{\text{CSRN}}$

$$\boxed{\text{CORN} = \text{CORJ} + \text{CSRN}}$$

$$y = Ce^{2x} + (x-1)e^{3x}$$

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CORN

(4) $y' + 3xy = x^2 \cos x$ $\quad + 3y$

$\underbrace{p(x) = 3x}_{\text{nie jest stałą}}$

RJ: $\frac{dy}{dx} = -3xy$

$$y = Ce^{-\int 3x} = \underbrace{C \cdot e^{-\frac{3}{2}x^2}}_{\text{CORJ}}$$

przew. $y = (ax^2 + bx + c)\cos x$

$y' = (2ax + b)\cos x - (ax^2 + bx + c)\sin x$

(4): $(2ax+b)\cos x - (ax^2 + bx + c)\sin x + 3x(ax^2+bx+c)\cos x$
$= x^2 \cos x$

$\vdots$

$y = ax^2 \cos x$

$y' = 2ax\cos x - ax^2 \sin x$

$\underbrace{2ax\cos x - ax^2 \sin x}_{y'} + 3x \underbrace{ax^2 \cos x}_{y} = x^2 \cos x$

$\Downarrow$

$\cos x\,(2ax + 2ax^2) = x^2 \cos x$

$2a = 1 \wedge 2a = 0$

sprzeczność

(5) $y' + y = \underbrace{x^2 e^{3x} + x\sin x + x^3 + 2x - 1}_{q(x)}$

$p(x) = 1$

metoda przewidywań

1° CORJ

2° $\underbrace{y' + y = x^2 e^{3x}}_{RN_1} \Rightarrow$ CSRN$_1$

RN_1

3e $\underbrace{y' + y = x \sin x}_{RN_2} \Rightarrow CSRN_2$

4e $y' + y = x^3 + 2x - 1 \Rightarrow CSRN_3$

$$\underline{CORN = CORJ + CSRN_1 + CSRN_2 + CSRN_3}$$