Ćw. 4

czwartek, 25 marca 2021 08:13

Zad. 1.

Obliczyć długość łuku krzywej:

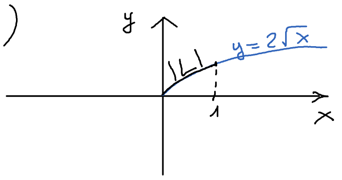
a) $y = 2\sqrt{x}$; $0 \le x \le 1$

b) $y = \ln x$; $\sqrt{3} \le x \le 2\sqrt{2}$.

Wzór na długość krzywej $y = f(x)$ dla $x \in \langle a, b \rangle$

$$|L| = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ad. a)



$$\frac{dy}{dx} = 2 \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$|L| = \int_0^1 \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx = \int_0^1 \sqrt{1 + \frac{1}{x}}\, dx$$

$$\int \sqrt{1 + \frac{1}{x}}\, dx = \int \sqrt{\frac{x+1}{x}}\, dx = \left\{ \begin{array}{l} \sqrt{x+1} = t \\ x+1 = t^2 \\ dx = 2t\,dt \\ \to x = t^2 - 1 \end{array} \right\} = \int \frac{t}{\sqrt{t^2-1}} 2t\,dt = 2\int t \frac{t}{\sqrt{t^2-1}}\, dt = (*)$$

$$\boxed{\text{Wzór: } \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C}$$

$$\int \frac{t}{\sqrt{t^2-1}}\, dt = \left\{ \begin{array}{l} \sqrt{t^2-1} = v \\ t^2 - 1 = v^2 \\ 2t\,dt = 2v\,dv \end{array} \right\} = \int \frac{t}{v} \cdot \frac{2v\,dv}{2t} = \int dv = v = \sqrt{t^2-1} + C \qquad dt = \frac{v\,dv}{t}$$

$$\int t \frac{t}{\sqrt{t^2-1}}\, dt = \left\{ \begin{array}{ll} f = t & g' = \frac{1}{2}\frac{2t}{2\sqrt{t^2-1}} \\ f' = 1 & g = \sqrt{t^2-1} \end{array} \right\} =$$

zwinięcie $\frac{1}{2} \cdot 2\sqrt{t^2-1} = \sqrt{t^2-1}$

$$= t\sqrt{t^2-1} - \int \sqrt{t^2-1}\, dt = t\sqrt{t^2-1} - \int \frac{t^2-1}{\sqrt{t^2-1}}\, dt = t\sqrt{t^2-1} - \int \frac{t^2}{\sqrt{t^2-1}}dt + \int \frac{dt}{\sqrt{t^2-1}}$$

Przenosimy $\int \frac{t^2}{\sqrt{t^2-1}} dt$ na lewą stronę równania

$$2\int \frac{t^2}{\sqrt{t^2-1}}\, dt = t\sqrt{t^2-1} + \int \frac{dt}{\sqrt{t^2-1}}$$

$$\boxed{\text{Wzór: } \int \frac{dx}{\sqrt{x^2+k}} = \ln|x + \sqrt{x^2+k}| + C}$$

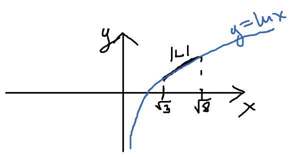
$$t^2 - 1 \ne 1 - t^2$$

$$\int \frac{t^2}{\sqrt{t^2-1}}\, dt = \frac{1}{2} t\sqrt{t^2-1} + \frac{1}{2}\ln|t + \sqrt{t^2-1}| + C$$

$$(*) = t\sqrt{t^2-1} + \ln|t + \sqrt{t^2-1}| + C = \sqrt{x+1}\cdot\sqrt{x} + \ln|\sqrt{x+1} + \sqrt{x}| + C = \sqrt{x^2+x} + \ln|\sqrt{x+1} + \sqrt{x}| + C$$

$$|L| = \left[\sqrt{x^2+x} + \ln|\sqrt{x+1} + \sqrt{x}|\right]\Big|_0^1 = \sqrt{2} + \ln|\sqrt{2}+1| - 0 - \underbrace{\ln 1}_{0} = \sqrt{2} + \ln(\sqrt{2}+1)$$

Ad. b)



$$\frac{dy}{dx} = \frac{1}{x}$$

$$|L| = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + \frac{1}{x^2}}\, dx$$

$$\int \sqrt{1 + \frac{1}{x^2}}\, dx = \int \sqrt{\frac{x^2+1}{x^2}}\, dx = \int \frac{\sqrt{x^2+1}}{x}\, dx = \left\{ \begin{array}{l} \sqrt{x^2+1} = t \\ x^2 + 1 = t^2 \\ x\,dx = t\,dt \\ \Downarrow \\ dx = \frac{t\,dt}{x} \\ \to x^2 = t^2 - 1 \end{array} \right\} = \int \frac{t}{x} \cdot \frac{t\,dt}{x} = \int \frac{t^2\,dt}{x^2} = \int \frac{t^2}{t^2-1}\, dt =$$

$$= \int \frac{(t^2-1)+1}{t^2-1}\, dt = \int \frac{\cancel{t^2-1}}{\cancel{t^2-1}}\, dt + \int \frac{1}{t^2-1}\, dt = \int dt + \underbrace{\int \frac{1}{(t-1)(t+1)}\, dt}_{\text{Ŋ}} = t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= \int \frac{(t^2-1)+1}{t^2-1}\,dt = \int \frac{\cancel{t^2-1}}{\cancel{t^2-1}}\,dt + \int \frac{1}{t^2-1}\,dt = \int dt + \underbrace{\int \frac{1}{(t-1)(t+1)}\,dt}_{J} = t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= \sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

J:

$$\frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} \quad \Big| \cdot (t-1)(t+1)$$

$$1 = A(t+1) + B(t-1)$$

$$t=-1 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$t = 1 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$J = \int \frac{\frac{1}{2}}{t-1}\,dt - \int \frac{\frac{1}{2}}{t+1}\,dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C =$$

$$= \frac{1}{2}\left(\ln|t-1| - \ln|t+1|\right) + C =$$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$|L| = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1+\frac{1}{x^2}}\,dx = \left[\sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|\right]\Bigg|_{\sqrt{3}}^{\sqrt{8}} = \left(3 + \frac{1}{2}\ln\frac{2}{4}\right) - \left(2 + \underline{\frac{1}{2}\ln\frac{1}{3}}\right) = 1 + \frac{1}{2}\left(\ln\frac{1}{2} - \ln\frac{1}{3}\right) =$$

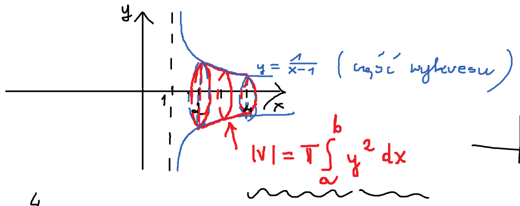
$$= 1 + \frac{1}{2}\ln\frac{3}{2}$$

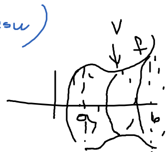
Zad. 2.

Obliczyć objętość bryły powstałej przez obrót krzywej a) $y = \sin x$ dookoła osi Ox dla $x \in \langle 0, \pi \rangle$

b) $y = \frac{1}{x-1}$ dookoła osi Oy dla $x \in \langle 2, 4 \rangle$

Ad b)





$$|V| = \pi \int_a^b y^2\,dx$$

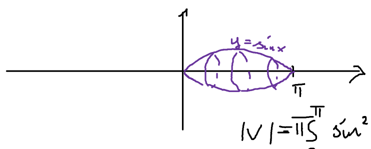
$$|V| = \pi \int_2^4 \frac{1}{(x-1)^2}\,dx$$

$$\int \frac{1}{(x-1)^2}\,dx = \left\{ \begin{array}{l} x-1=t \\ dx = dt \end{array} \right\} = \int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = -t^{-1} + C =$$

$$= -\frac{1}{t} + C = -\frac{1}{x-1} + C$$

$$|V| = \pi\left[-\frac{1}{x-1}\right]\Bigg|_2^4 = \pi\left(-\frac{1}{3}+1\right) = \frac{2}{3}\pi\,[j^3]$$

Ad. a) $y = \sin x$, $x \in \langle 0, \pi \rangle$



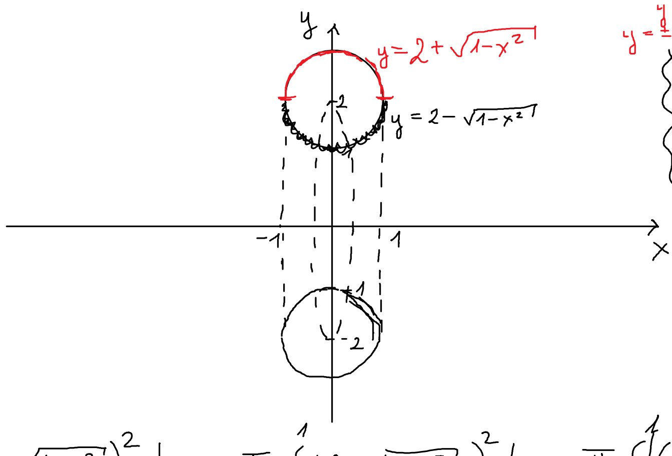
$$|V| = \pi \int_0^{\pi} \sin^2 x\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]\Bigg|_0^{\pi} = \pi\left(\frac{\pi}{2} - \frac{1}{4}\underbrace{\sin 2\pi}_{0} - 0\right)$$

$$|V| = \frac{\pi^2}{2}\,[j^3]$$

Zad. 3.

Obliczyć objętość bryły powstałej przez obrót koła $x^2 + (y-2)^2 \le 1$ dookoła osi Ox.

$$(y-2)^2 = 1-x^2$$
$$y-2 = \pm\sqrt{1-x^2}$$
$$y = \pm\sqrt{1-x^2} + 2$$



Uwaga!

Środek koła porusza się po okręgu o promieniu $R = 2$ (długość trasy $= 4\pi$)

Objętość bryły jest równa iloczynowi 4π przez pole koła.

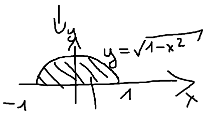
Tutaj $|V| = 4\pi \cdot \pi = 4\pi^2\,[j^3]$

$$|V| = \pi \int_{-1}^{1} \left(2+\sqrt{1-x^2}\right)^2 dx - \pi \int_{-1}^{1} \left(2-\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} \left[\left(2+\sqrt{1-x^2}\right)^2 - \left(2-\sqrt{1-x^2}\right)^2\right] dx =$$

$$|V| = \pi \int_{-1}^{1} \left(2+\sqrt{1-x^2}\right)^2 dx - \pi \int_{-1}^{1} \left(2-\sqrt{1-x^2}\right)^2 dx = \pi \int_{-1}^{1} \left[\left(2+\sqrt{1-x^2}\right)^2 - \left(2-\sqrt{1-x^2}\right)^2\right] dx =$$

$$= \pi \int_{-1}^{1} \left[\left(4+4\sqrt{1-x^2}+1-x^2\right) - \left(4-4\sqrt{1-x^2}+1-x^2\right)\right] dx = \pi \int_{-1}^{1} 8\sqrt{1-x^2}\, dx =$$

$$= 8\pi \underbrace{\int_{-1}^{1} \sqrt{1-x^2}\, dx} = 8\pi \cdot \frac{\pi}{2} = \underline{\underline{4\pi^2}}$$



$P = \frac{\pi}{2}$