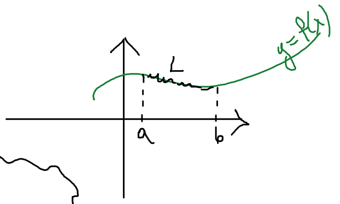
Wzór na długość łuku krzywej

$$y=f(x), \quad a\le x\le b$$

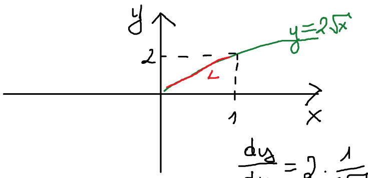


$$\boxed{|L| = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Zad. 1. Oblicz długość łuku krzywej a) $y=2\sqrt{x}$, $0\le x\le 1$,

b) $y=\ln x$, $\sqrt{3}\le x\le 2\sqrt{2}$.

Ad. a)



$$\frac{dy}{dx} = 2\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Wzór

$$\boxed{\int\frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}+C}$$

Wzór

$$\boxed{\int\frac{1}{\sqrt{x^2-1}}dx = \ln|x+\sqrt{x^2-1}|+C}$$

$$|L| = \int_0^1 \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx = \int_0^1\sqrt{1+\frac{1}{x}}\,dx = \dots$$

Obliczymy całkę nieoznaczoną.

$$\int\sqrt{1+\frac{1}{x}}\,dx = \int\sqrt{\frac{x+1}{x}}\,dx = \int\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = \left\{\begin{array}{l}\sqrt{x+1}=t\\ x+1=t^2\\ dx=2t\,dt\\ x=t^2-1\end{array}\right\} =$$

$$= \int\frac{t}{\sqrt{t^2-1}}\cdot 2t\,dt = 2\int t\,\frac{t}{\sqrt{t^2-1}}\,dt = \left\{\begin{array}{ll} f=t & g'=\frac{1}{2}\frac{2t}{\sqrt{t^2-1}}\\ f'=1 & g=\frac{1}{2}\cdot 2\sqrt{t^2-1}=\sqrt{t^2-1}\end{array}\right\} = 2\left(t\sqrt{t^2-1}-\int\sqrt{t^2-1}\,dt\right)=$$

$$=2t\sqrt{t^2-1} - 2\int\frac{t^2-1}{\sqrt{t^2-1}}\,dt = 2t\sqrt{t^2-1} - 2\int\frac{t^2}{\sqrt{t^2-1}}\,dt + 2\int\frac{1}{\sqrt{t^2-1}}\,dt =$$

$$2\int\frac{t^2}{\sqrt{t^2-1}}\,dt = 2t\sqrt{t^2-1} - 2\int\frac{t^2}{\sqrt{t^2-1}}\,dt + 2\ln|t+\sqrt{t^2-1}|+C \quad \Big|:2$$

$$2\int\frac{t^2}{\sqrt{t^2-1}}\,dt = t\sqrt{t^2-1}+\ln|t+\sqrt{t^2-1}|+C \quad \Big|\cdot 2$$

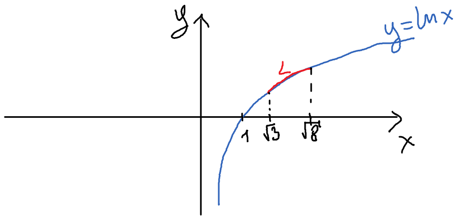
$$\int\frac{t^2}{\sqrt{t^2-1}}\,dt = \frac{1}{2}t\sqrt{t^2-1}+\frac{1}{2}\ln|t+\sqrt{t^2-1}|+C$$

Wracamy do podstawienia

$$\int\sqrt{1+\frac{1}{x}}\,dx = \frac{1}{2}\sqrt{x+1}\sqrt{x} + \frac{1}{2}\ln|\sqrt{x+1}+\sqrt{x}|+C = \frac{1}{2}\sqrt{x^2+x}+\frac{1}{2}\ln|\sqrt{x+1}+\sqrt{x}|+C$$

$$|L| = \int_0^1\sqrt{1+\frac{1}{x}}\,dx = \left[\frac{1}{2}\sqrt{x^2+x}+\frac{1}{2}\ln|\sqrt{x+1}+\sqrt{x}|\right]\Big|_0^1 = \frac{1}{2}\sqrt{2}+\frac{1}{2}\ln(\sqrt{2}+1) - 0 - \frac{1}{2}\underbrace{\ln 1}_{0} = \frac{1}{2}\sqrt{2}+\frac{1}{2}\ln(1+\sqrt{2})$$

Ad b) $y=\ln x$, $x\in\langle\sqrt{3}, 2\sqrt{2}\rangle$



$$\frac{dy}{dx} = \frac{1}{x}$$

$$|L| = \int_{\sqrt{3}}^{\sqrt{8}}\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx = \int_{\sqrt{3}}^{\sqrt{8}}\sqrt{1+\frac{1}{x^2}}\,dx = \int_{\sqrt{3}}^{\sqrt{8}}\sqrt{\frac{x^2+1}{x^2}}\,dx = \int_{\sqrt{3}}^{\sqrt{8}}\frac{\sqrt{x^2+1}}{x}\,dx = \dots$$

$$\int\frac{\sqrt{x^2+1}}{x}\,dx = \left\{\begin{array}{l}\sqrt{x^2+1}=t\\ x^2+1=t^2\end{array}\right\} = \int\frac{t}{x}\cdot\frac{t\,dt}{x} = \int\frac{t^2\,dt}{x^2} = \int\frac{t^2}{t^2-1}\,dt = \int\frac{(t^2-1)+1}{t^2-1}\,dt=$$

$$\int \frac{\sqrt{x^2+1}}{x}\,dx = \left\{\begin{array}{l} \sqrt{x^2+1} = t \\ x^2+1 = t^2 \\ x\,dx = t\,dt \\ dx = \frac{t\,dt}{x} \\ x^2 = t^2-1 \end{array}\right\} = \int \frac{t}{x}\cdot\frac{t\,dt}{x} = \int \frac{t^2\,dt}{x^2} = \int \frac{t^2}{t^2-1}\,dt = \int \frac{(t^2-1)+1}{t^2-1}\,dt =$$

$$= \int dt + \underbrace{\int \frac{1}{t^2-1}\,dt}_{J} = t + J = (*)$$

$$J = \int \frac{1}{(t-1)(t+1)}\,dt$$

$$\frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1}, \quad A,B \in \mathbb{R}$$

$$1 = A(t+1) + B(t-1)$$

dla $t = -1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

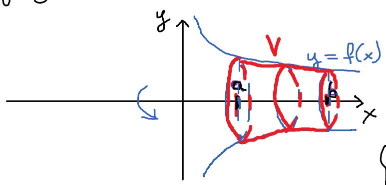
dla $t = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$$J = \int \frac{\frac{1}{2}}{t-1}\,dt - \int \frac{\frac{1}{2}}{t+1}\,dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C = \frac{1}{2}\ln\frac{|t-1|}{|t+1|} + C$$

$$(*) = t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C$$

$$|L| = \left[\sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|\right]\Bigg|_{\sqrt{3}}^{\sqrt{8}} = \left(3 + \frac{1}{2}\ln\frac{2}{4}\right) - \left(2 + \frac{1}{2}\ln\frac{1}{3}\right) = 1 + \frac{1}{2}\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \underline{\underline{1 + \frac{1}{2}\ln\frac{3}{2}}}$$

Objętość bryły V



$$|V| = \pi\int_a^b y^2\,dx$$

Zad. 2.

Obliczyć objętość bryły powstałej przez obrót dookoła osi Ox, krzywej

a) $y = \frac{1}{x-1}$ dla $x \in \langle 2, 4\rangle$

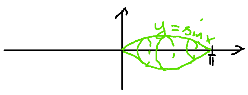
b) $y = \sin x$ dla $x \in \langle 0, \pi\rangle$

Ad a) $$|V| = \pi\int_2^4 \left(\frac{1}{x-1}\right)^2 dx = \pi\left[-\frac{1}{x-1}\right]\Bigg|_2^4 = \pi\left(-\frac{1}{3}+1\right) = \underline{\underline{\frac{2}{3}\pi}}\,[j^3]$$

$$\int \frac{1}{(x-1)^2}\,dt = -\frac{1}{x-1} + C$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

Ad. b) $$|V| = \pi\int_0^\pi \sin^2 x\,dx =$$



$$\begin{cases} \sin^2 x + \cos^2 x = 1 \\ \cos^2 x - \sin^2 x = \cos 2x \end{cases}$$

$$- \begin{cases} \int \sin^2 x\,dx + \int \cos^2 x\,dx = x \\ \int \cos^2 x\,dx - \int \sin^2 x\,dx = \frac{1}{2}\sin 2x \end{cases}$$

$$2\int \sin^2 x\,dx = x - \frac{1}{2}\sin 2x + C$$

$$\boxed{\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C}$$

$$|V| = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]\Bigg|_0^\pi = \pi\left[\left(\frac{1}{2}\pi - \frac{1}{4}\underbrace{\sin 2\pi}_{0}\right) - 0\right] = \underline{\underline{\frac{1}{2}\pi^2}}\,[j^3]$$

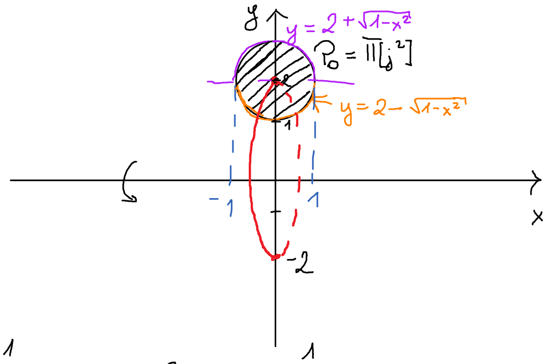
Zad. 3.

Obliczyć objętość bryły powstałej przez obrót dookoła osi ox koła $x^2 + (y-2)^2 \le 1$.

$K(S(0,2), r = 1)$

Zad. 3.
Obliczyć objętości bryły powstałej przez obrót dookoła osi ox koła $x^2+(y-2)^2 \le 1$.
$K(S(0,2), r=1)$

$(y-2)^2 = 1-x^2$
$y-2 = \pm\sqrt{1-x^2}$
$y = \pm\sqrt{1-x^2}+2$



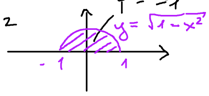
Otrzymany torus
środek koła pokonuje „trasę" 4π
$$|V| = \underbrace{4\pi \cdot \pi}_{\text{pole koła}} = 4\pi^2 [j^3]$$

$$|V| = \pi\int_{-1}^{1}(2+\sqrt{1-x^2})^2dx - \pi\int_{-1}^{1}(2-\sqrt{1-x^2})^2dx = \pi\int_{-1}^{1}\Big[\left(4+4\sqrt{1-x^2}+1-x^2\right)-\left(4-4\sqrt{1-x^2}\right)+1-x^2\Big]dx =$$

$$= \pi\int_{-1}^{1} 8\sqrt{1-x^2}\,dx = 8\pi \underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx} = 8\pi\cdot\frac{\pi}{2} = 4\pi^2 [j^3]$$

$y=\sqrt{1-x^2} \quad |^2$
$y^2 = 1-x^2$
$x^2+y^2=1$
$O(S(0,0), r=1)$

$P = \int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{\pi}{2}$



Całki niewłaściwe

$$\int_{-\infty}^{a} \qquad \int_{a}^{\infty} \qquad \text{lub} \qquad \int_{a}^{b} f(x)dx$$
$a \notin D_f$

Zadanie 4
Obliczyć całki niewłaściwe.

(1) $$\int_1^\infty \frac{dx}{x^2-6x+13} = \lim_{\beta\to\infty}\int_1^\beta \frac{dx}{x^2-6x+13}$$

$$\int\frac{dx}{x^2-6x+13} = \int\frac{dx}{(x-3)^2+4} = \left\{\begin{array}{l} x-3=2t \\ dx = 2dt \\ t=\frac{x-3}{2}\end{array}\right\} = \int\frac{2\,dt}{4t^2+4} = \frac{1}{2}\int\frac{dt}{t^2+1} = \frac{1}{2}\,\mathrm{arctg}\,t + C = \frac{1}{2}\,\mathrm{arctg}\frac{x-3}{2}+C$$

$$\int_1^\infty\frac{dx}{x^2-6x+13} = \lim_{\beta\to\infty}\left[\frac{1}{2}\,\mathrm{arctg}\frac{x-3}{2}\right]\Bigg|_1^\beta = \lim_{\beta\to\infty}\left[\frac{1}{2}\,\mathrm{arctg}\underbrace{\frac{\beta-3}{2}}_{\to\infty}-\frac{1}{2}\,\mathrm{arctg}(-1)\right] = \frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}+\frac{\pi}{8} = \frac{3}{8}\pi$$

$\mathrm{tg}(-\frac{\pi}{4}) = -1$
$\mathrm{arctg}(-1) = -\frac{\pi}{4}$

(2) $$\int_{-\infty}^{-\frac{1}{2}}\frac{dx}{x^2+x+1} = \lim_{A\to-\infty}\int_A^{-\frac{1}{2}}\frac{dx}{x^2+x+1} = (*)$$

$$\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \left\{\begin{array}{l} x+\frac{1}{2} = \frac{\sqrt{3}}{2}t \\ dx = \frac{\sqrt{3}}{2}dt \\ t = \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\end{array}\right\} = \int\frac{\frac{\sqrt{3}}{2}dt}{\frac{3}{4}t^2+\frac{3}{4}} = \frac{\sqrt{3}}{2}\cdot\frac{4}{3}\int\frac{dt}{} \quad \frac{}{3}\,\mathrm{arctg}\,t + C =$$
$$= \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$(*) = \lim_{A\to-\infty}\left[\frac{2\sqrt{3}}{3}\,\mathrm{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]\Bigg|_A^{-\frac{1}{2}} =$$

$$= \lim_{A\to-\infty}\left[\frac{2\sqrt{3}}{3}\underbrace{\mathrm{arctg}\,0}_{0} - \frac{2\sqrt{3}}{3}\,\mathrm{arctg}\underbrace{\frac{A+\frac{1}{2}}{\frac{\sqrt{3}}{2}}}_{\to-\infty}\right] = \frac{2\sqrt{3}}{3}\cdot\frac{\pi}{2} = \frac{\sqrt{3}\pi}{3}$$
$\to -\frac{\pi}{2}$

(3) $$\int^{\infty}\frac{dx}{x^2+2x+2} = \int^{a}\frac{dx}{x^2+2x+2} + \int^{\infty}\frac{dx}{x^2+2x+2}$$

$A \to -\infty$

(3) $$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{a} \frac{dx}{x^2+2x+2} + \int_{a}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2} =$$

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \operatorname{arctg}(x+1) + C$$

$$= \lim_{A\to-\infty} \operatorname{arctg}(x+1)\Big|_A^0 + \lim_{B\to\infty} \operatorname{arctg}(x+1)\Big|_0^B =$$

$$= \lim_{A\to-\infty}\big(\operatorname{arctg}1 - \underbrace{\operatorname{arctg}(A+1)}_{\to -\frac{\pi}{2}}\big) + \lim_{B\to\infty}\big(\underbrace{\operatorname{arctg}(B+1)}_{\to \frac{\pi}{2}} - \operatorname{arctg}1\big) =$$

$$= \operatorname{arctg}1 + \frac{\pi}{2} + \frac{\pi}{2} - \operatorname{arctg}1 = \underline{\underline{\pi}}$$

(4) $$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{A\to\infty}\left[-\arcsin\frac{1}{x}\right]_1^A =$$

$$= \lim_{A\to\infty}\Big(\underbrace{-\arcsin\underbrace{\frac{1}{A}}_{\to 0}}_{\to 0} + \underbrace{\arcsin 1}_{\frac{\pi}{2}}\Big) = \underline{\underline{\frac{\pi}{2}}}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \left\{\frac{1}{x} = t\right.$$

$$\int \frac{dx}{x\sqrt{x^2(1-\frac{1}{x^2})}} = \int \frac{dx}{x^2\sqrt{1-\frac{1}{x^2}}} = \left\{\begin{matrix}\frac{1}{x}=t \\ \frac{1}{x^2}dx = -dt\end{matrix}\right\} =$$

$$= \int \frac{-dt}{\sqrt{1-t^2}} = -\arcsin t + C = -\arcsin\frac{1}{x} + C$$

(5) $$\int_0^\infty e^{-x}\sin x\,dx =$$

$$= \lim_{A\to\infty}\int_0^A e^{-x}\sin x\,dx =$$

$$= \lim_{A\to\infty}\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^A =$$

$$= \lim_{A\to\infty}\Big[\underbrace{-\frac{1}{2}e^{-A}}_{e^{-\infty}\to 0}\underbrace{(\sin A + \cos A)}_{\text{ograniczone}} + \frac{1}{2}\underbrace{e^0}_{1}(\underbrace{\sin 0}_{0} + \underbrace{\cos 0}_{1})\Big] = \underline{\underline{\frac{1}{2}}}$$

iloczyn ciągu ograniczonego i ciągu $\to 0$ jest zbieżny do 0

$$\int e^{-x}\sin x\,dx = \left\{\begin{matrix} f = e^{-x} & g' = \sin x \\ f' = -e^{-x} & g = -\cos x\end{matrix}\right\} = -e^{-x}\cos x - \int e^{-x}\cos x\,dx =$$

$$\left\{\begin{matrix} f = e^{-x} & g' = \cos x \\ f' = -e^{-x} & g = \sin x\end{matrix}\right\}$$

$$= -e^{-x}\cos x - \left(e^{-x}\sin x + \int e^{-x}\sin x\,dx\right) =$$

$$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$$

$$2\int e^{-x}\sin x\,dx = -e^{-x}(\cos x + \sin x) \quad |:2$$

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$$

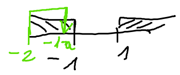
Zadanie 5

oblicz całki niewłaściwe:

(1) $$\int_{-2}^{-1} \frac{dx}{x\sqrt{x^2-1}},$$

$$f(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$D_f: \begin{cases} x^2-1>0 \\ x\neq 0\end{cases}$$



$(-\infty, -1) \cup (1, \infty)$

$$= \lim_{a\to 0}\int_{-2}^{-1-a} \frac{dx}{x\sqrt{x^2-1}} = \lim_{a\to 0}\left[-\arcsin\frac{1}{x}\right]_{-2}^{-1-a} =$$

$$= \lim_{a\to 0}\Big[-\arcsin\underbrace{\frac{1}{-1-a}}_{\to -1} + \arcsin\left(\frac{1}{-2}\right)\Big] = -\arcsin(-1) + \arcsin\left(-\frac{1}{2}\right) =$$

$$= -\left(-\frac{\pi}{2}\right) - \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{6} = \underline{\underline{\frac{\pi}{3}}}$$