Zad. 1.

Pierwiastki 4 stopnia z liczby $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\{\omega_0, \omega_1, \omega_2, \omega_3\}$

$$\omega_k = \underbrace{\sqrt[n]{|z|}}_{\text{pierw. arytmetyczny}} \left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right)$$

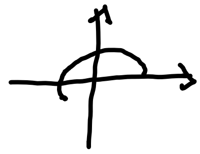
$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$a = -\frac{1}{2}$

$b = -\frac{\sqrt{3}}{2}$

$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\cos\varphi = -\frac{1}{2}$, $\sin\varphi = -\frac{\sqrt{3}}{2}$ → III ćw. → $\varphi = \pi + \alpha$, $\alpha = \frac{\pi}{3}$



$\varphi = \pi + \frac{\pi}{3} = \frac{4}{3}\pi$

$\omega_0 = \sqrt[4]{1}\left(\cos\frac{\frac{4}{3}\pi + 0}{4} + i\sin\frac{\frac{4}{3}\pi+0}{4}\right) = \cos\underbrace{\frac{\pi}{3}}_{\varphi_0} + i\sin\underbrace{\frac{\pi}{3}}_{\varphi_0} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\omega_1 = \sqrt[4]{1}\left(\cos\frac{\frac{4}{3}\pi + 2\pi}{4} + i\sin\frac{\frac{4}{3}\pi+2\pi}{4}\right) = \cos\underbrace{\frac{5}{6}\pi}_{\varphi_1} + i\sin\underbrace{\frac{5}{6}\pi}_{\varphi_1} = \cos(\pi - \frac{\pi}{6}) + i\sin(\pi - \frac{\pi}{6}) =$

$\frac{\frac{4}{3}\pi + \frac{6}{3}\pi}{4} = \frac{10}{12}\pi = \frac{5}{6}\pi$

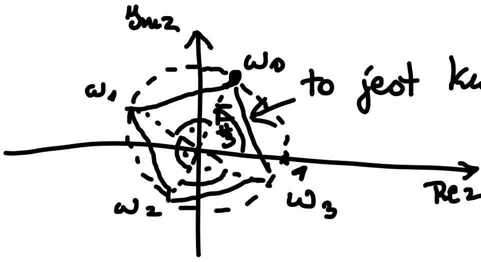
$= -\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\frac{2\pi}{4} = \frac{\pi}{2}$

$\underset{\varphi_0}{\frac{\pi}{3}} \xrightarrow{+\frac{\pi}{2}} \underset{\varphi_1}{\frac{5}{6}\pi} \xrightarrow{+\frac{\pi}{2}} \underset{\varphi_2}{\frac{8}{6}\pi} \xrightarrow{+\frac{\pi}{2}} \frac{11}{6}\pi$

$\omega_2 = \cos\frac{8}{6}\pi + i\sin\frac{8}{6}\pi = \cos(\pi + \frac{\pi}{3}) + i\sin(\pi + \frac{\pi}{3}) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\omega_3 = \cos(2\pi - \frac{\pi}{6}) + i\sin(2\pi - \frac{\pi}{6}) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.



to jest kwadrat ;)

Zad. 2.

$\sqrt[6]{27}$ $\{\omega_0, \omega_1, \ldots, \omega_5\}$

$a = 27$ $b = 0$

$z = 27 \Rightarrow |z| = 27$

$\varphi = 0$

$\sqrt[6]{3^3} = (3^3)^{\frac{1}{6}} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}} = \sqrt{3}$

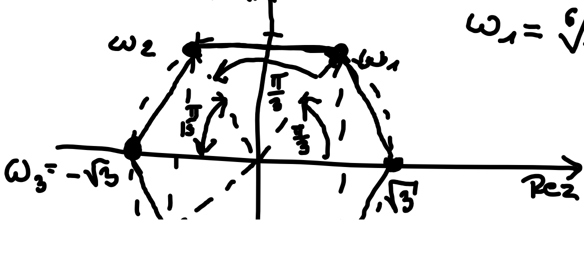
$\omega_0 = \sqrt{3}$

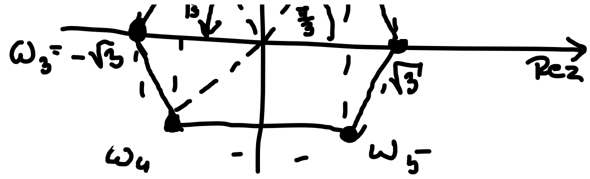
$\omega_0 = \sqrt[6]{27}(\underbrace{\cos 0}_{1} + i\underbrace{\sin 0}_{0}) = \sqrt[6]{27} = \sqrt{3}$

$\omega_1 = \sqrt[6]{27}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \sqrt{3}(\frac{1}{2} + \frac{\sqrt{3}}{2}i) =$

$\omega_1 = \frac{\sqrt{3}}{2} + \frac{3}{2}i$

$\omega_2 = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$





$\omega_2 = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

$\omega_3 = -\sqrt{3}$

$\frac{2\pi}{6} = \frac{\pi}{3} \Rightarrow$ argumenty pierwiastków różnią się o $\frac{\pi}{3}$ (kolejne)

$\omega_4 = -\frac{\sqrt{3}}{2} - \frac{3}{2}i$

$\omega_5 = \frac{\sqrt{3}}{2} - \frac{3}{2}i$

Zad. 3.

Rozwiązać równanie $z^2 + \underset{b}{3}z + \underset{c}{3-i} = 0$ w zbiorze liczb zespolonych.

Obliczamy $\Delta = 3^2 - 4(3-i) = 9 - 12 + 4i = -3 + 4i$

$\sqrt{\Delta} = ?$

$\sqrt{-3+4i}$

Niech: $\sqrt{-3+4i} = a + bi$; $a, b \in \mathbb{R}$

$(a+bi)^2 = -3+4i$

$a^2 + 2abi + b^2i^2 = -3+4i$

$(a^2 - b^2) + 2abi = -3 + 4i$

$\begin{cases} a^2 - b^2 = -3 \\ 2ab = 4 \Rightarrow b = \frac{4}{2a} = \frac{2}{a} \end{cases}$

$z = -3+4i$

$|z| = \sqrt{9+16} = 5$

$\cos\varphi = -\frac{3}{5}$

$\sin\varphi = \frac{4}{5}$

Inna metoda wyznaczania pierw. kwadratowego

$\begin{cases} b = \frac{2}{a} \\ a^2 - b^2 = -3 \end{cases}$

$\begin{cases} b = \frac{2}{a} \\ a^2 - \frac{4}{a^2} = -3 \quad / \cdot a^2 \end{cases}$

$\begin{cases} b = \frac{2}{a} \\ a^4 - 4 = -3a^2 \quad (*) \end{cases}$

$(*) \quad a^4 + 3a^2 - 4 = 0$

↖ r-nie dwukwadratowe

Niech $t = a^2$, $t > 0$

$t^2 + 3t - 4 = 0$

$\Delta_t = 9 + 16 = 25 \quad \sqrt{\Delta_t} = 5$

$t_1 = \frac{-3-5}{2} < 0$

$t_2 = \frac{-3+5}{2} = 1 \Rightarrow a^2 = 1$

$\begin{cases} a = 1 \\ b = 2 \end{cases} \vee \begin{cases} a = -1 \\ b = -2 \end{cases}$

$\sqrt{\Delta} = \sqrt{-3+4i} = \{1+2i; -1-2i\}$

$z = \frac{-3-(1+2i)}{2} = \frac{-4-2i}{2} = -2-i$

$$z_1 = \frac{-3-(1+2i)}{2} = \frac{-4-2i}{2} = -2-i$$

$$z_2 = \frac{-3+1+2i}{2} = \frac{-2+2i}{2} = -1+i$$

Zad. 4.

Rozwiązać r-nie $z^2 - z + 1 = 0$ w zbiorze $z \in \mathbb{C}$.

$\Delta = -3 \qquad \sqrt{\Delta} = \{-\sqrt{3}i, \sqrt{3}i\}$

$\sqrt{-3} = \sqrt{3i^2} = \sqrt{3}i$

$z_1 = \frac{1+\sqrt{3}i}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \frac{1-\sqrt{3}i}{2} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Uwaga! z_1 i z_2 są sprzężone

Wielomian o współczynnikach rzeczywistych:
Jeśli z jest pierwiastkiem, to $\bar{z}$ też jest pierwiastkiem

Zad. 5

Rozwiązać równanie $z^4 + 8z^2 + 15 = 0, \; z \in \mathbb{C}$

Niech $z^2 = t, \; t \in \mathbb{C}$

$t^2 + 8t + 15 = 0$

$\Delta = 64 - 4 \cdot 15 = 4$

$\sqrt{\Delta} = 2$

$t_1 = \frac{-8-2}{2} = -5 \Rightarrow z^2 = -5 \Rightarrow z = -\sqrt{5}i \vee z = \sqrt{5}i$

$t_2 = \frac{-8+2}{2} = -3 \Rightarrow z^2 = -3 \Rightarrow z = -\sqrt{3}i \vee z = \sqrt{3}i$

Zad. 6.

Rozłożyć na czynniki liniowe wielomian zespolony $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6 = 0$

$1, -1, 2, -2, 3, -3, 6, -6$

	1	-2	-4	4	-5	6
1	1	-1	-5	-1	-6	0
-2	1	-3	1	-3	0	

$(x-1)(x^4 - x^3 - 5x^2 - x - 6) = 0$

$(x-1)(x+2)(x^3 - 3x^2 + x - 3) = 0$

$x^2(x-3) + (x-3) = 0$

$(x-3)(x^2+1) = 0$

$(x-1)(x+2)(x-3)(x^2+1) = 0$

$x = i \vee x = -i$

$(x-1)(x+2)(x-3)(x-i)(x+i) = 0$

Zad. 7.

Wiedząc, że $z_1 = 1 + 2i$ jest pierwiastkiem wielomianu zespolonego $W(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$, wyznaczyć pozostałe pierwiastki.

$W(z)$ - ma współczynniki rzeczywiste, więc $z_2 = 1 - 2i$

$W(z)$ dzieli się przez $(z - z_1)$ i $(z - z_2)$

$W(z)$ dzieli się przez $(z-z_1)$ i $(z-z_2)$

$$(z-(1+2i))(z-(1-2i)) = z^2 - z(1-2i) - z(1+2i) + (1+2i)(1-2i) =$$

$$= z^2 - z(1-2i+1+2i) + 1 - 2^2 i^2 = z^2 - 2z + 5$$

$$(z^4 - 2z^3 + 6z^2 - 2z + 5) : (z^2 - 2z + 5) = z^2 + 1$$

$$\underline{-z^4 + 2z^3 - 5z^2}$$

$$= \quad =$$

$$z^2 - 2z + 5$$

$$\underline{-z^2 + 2z - 5}$$

$$= \quad = \quad =$$

$$\Downarrow$$

$$z^2 = -1$$

$$z_3 = i \lor z_4 = -i$$

Zad. 8.

Rozłożyć na czynniki liniowe wielomian zespolony

$$W(z) = -z^4 + z^3 - z^2 + z$$

Szukamy pierwiastków wielomianu

$$-z^4 + z^3 - z^2 + z = 0$$

$$z(-z^3 + z^2 - z + 1) = 0$$

$$z(z^2(-z+1) + (-z+1)) = 0$$

$$z(1-z)(z^2+1) = 0$$

$$z(1-z)(z-i)(z+i) = 0$$

$$z = 0 \lor z = 1 \lor z = i \lor z = -i$$

Ad. zad. 7.

$$W(z) = z^4 - 2z^3 + 6z^2 - 2z + 5 \qquad z_1 = 1+2i$$

$(2i+1)(2i-1) = -4-1$

$(-1+2i)(1+2i) = (1+2i)($

z^4:	1	-2	6	-2	5
$1+2i$		$1+2i$	-5	$1+2i$	-5
z^3	1	$-1+2i$	1	$-1+2i$	0
$1-2i$		$1-2i$	0	$1-2i$	
z^2	1	0	1	0	

$$z^2 + 1 = 0$$