Ćwiczenia nr 13

środa, 2 czerwca 2021

R-nie Bernulliego

Postać ogólna : $\boxed{y' + p(x)\cdot y = q(x)\cdot y^{\alpha}}$

Uwaga!

$\alpha = 0 \Rightarrow$ r-nie liniowe

$\alpha = 1 \Rightarrow$ r-nie liniowe jednorodne

$y' + p(x)\cdot y = q(x)\cdot y$

$y' + y\underbrace{(p(x) - q(x))}=0$

Dla $\alpha \neq 0$ i $\alpha \neq 1$ stosujemy podstawienie :

$$\boxed{u = y^{1-\alpha}}$$

(1) $xy' - 4y = x^2\sqrt{y} \quad /:x \quad , \quad x\neq 0$

(*) $y' - \frac{4}{x}y = x\sqrt{y}$

$p(x) = -\frac{4}{x} \quad , \quad q(x) = x \quad , \quad \alpha = \frac{1}{2}$

Podstawienie

$$u = y^{\frac{1}{2}}$$

Różniczkujemy po x pamiętając, że u jest f-cją x

$$u' = \frac{1}{2}y^{-\frac{1}{2}}\cdot y'$$

Równanie (*) pomnożymy przez $\frac{1}{2}y^{-\frac{1}{2}}$:

$$\frac{1}{2}y^{-\frac{1}{2}}y' - \frac{1}{\cancel{2}}y^{-\frac{1}{2}}\cdot\frac{\cancel{4}}{x}\cdot y = \frac{1}{2}y^{-\frac{1}{2}}\cdot x\cdot y^{\frac{1}{2}}$$

$$\frac{1}{2} y^{-} y' - \cancel{\frac{1}{2}} y^{-2} \cdot \frac{\cancel{4}}{x} \cdot y = \frac{1}{2} \underbrace{y^{-2} \cdot x \cdot y^{2}}_{y^0 = 1}$$

$$\underbrace{\frac{1}{2} y^{-\frac{1}{2}} y'}_{u'} - \frac{2}{x} \underbrace{y^{\frac{1}{2}}}_{u} = \frac{1}{2} x$$

Teraz mamy RL: $u' - \frac{2}{x} u = \frac{1}{2} x$ ← r-nie liniowe niejednorodne

1° RJ: $u' - \frac{2}{x} u = 0$

$$\frac{du}{dx} = \frac{2}{x} u \quad \Big/ \cdot \frac{dx}{u}, \quad u \neq 0$$

$$\frac{du}{u} = \frac{2}{x} dx$$

$$\ln|u| = 2 \ln|x| + C$$

$$u = C x^2$$

CORJ

2° uzmiennianie stałej. $u = C(x) \cdot x^2$

wtedy $u' = C'(x) x^2 + 2x C(x)$

wstawiając u i u' do RL, mamy:

$$C'(x) x^2 + 2x C(x) - \frac{2}{\cancel{x}} C(x) x^{\cancel{2}} = \frac{1}{2} x$$

$$C'(x) x^2 = \frac{1}{2} x \implies C'(x) = \frac{1}{2x}$$

$$C(x) = \int \frac{1}{2x} dx = \frac{1}{2} \ln|x| + C$$

Rozwiązanie ogólne r-nia liniowego RL:

$$u = \left(\frac{1}{2} \ln|x| + C\right) x^2 \leftarrow \text{CORN}$$

$$u = \frac{1}{2} x^2 \ln|x| + C x^2$$

② Rozwiązać r-nie $\quad y^{\frac{1}{2}} = \frac{1}{2}x^2\ln|x| + Cx^2$

(*) $y' - \frac{\quad}{2(\cancel{x^2}y1)} = \frac{\cancel{2x}}{y}$, przy warunku początkowym $y(\sqrt{2}) = 0$ $\left(\begin{cases} x = \sqrt{2} \\ y = 0 \end{cases} \right)$.

$$p = -\frac{x}{2(x^2-1)}, \quad q = \frac{x}{2}, \quad \alpha = -1$$

Podstawienie: $\quad u = y^2 \qquad \boxed{u = y^{1-\alpha}}$

$$\Downarrow$$

$$u' = 2yy'$$

r-nie (*) mnożymy przez $2y$:

$$\underbrace{2yy'}\, - \frac{\cancel{2}xy^2}{\cancel{2}(x^2-1)} = x$$

RL: $\quad u' - \frac{x}{x^2-1}u = x$

1° RJ: $\quad u' - \frac{xu}{x^2-1} = 0$

$$\frac{du}{dx} = \frac{xu}{x^2-1} \quad \Big/ \cdot \frac{dx}{u}, \quad u \neq 0$$

$$\frac{du}{u} = \frac{x}{x^2-1}dx$$

$$\int\frac{du}{u} = \frac{1}{2}\int\frac{2x}{x^2-1}dx$$

$$\ln|u| = \frac{1}{2}\ln|x^2-1| + \underset{\ln C}{C}$$

$$u = C\sqrt{x^2-1}, \qquad x \in (-\infty, -1\rangle \cup \langle 1, \infty)$$

wL

$$\underbrace{u = C\sqrt{x^2-1}}_{CORJ}, \qquad x \in (-\infty,-1\rangle \cup \langle 1,\infty)$$

2° uzmienniamy stałą C.

$$u = C(x)\cdot\sqrt{x^2-1}$$

$$u' = C'(x)\sqrt{x^2-1} + C(x)\cdot\frac{x}{\sqrt{x^2-1}}$$

$$RL \to \underbrace{C'(x)\sqrt{x^2-1} + \cancel{\frac{x\,C(x)}{\sqrt{x^2-1}}}}_{u'} - \cancel{\frac{x}{x^2-1}\cdot \underbrace{C(x)\cdot\sqrt{x^2-1}}_{u}} = x$$

$$\frac{x\cdot C(x)\cdot\sqrt{x^2-1}}{x^2-1} = \frac{x\,C(x)\,\cancel{\sqrt{x^2-1}}}{\left(\sqrt{x^2-1}\right)^{2}} = \frac{x\,C(x)}{\sqrt{x^2-1}}$$

$$C'(x)\sqrt{x^2-1} = x \Rightarrow C'(x) = \frac{x}{\sqrt{x^2-1}}$$

$$C(x) = \int\frac{x}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + C$$

Rozwiązanie ogólne r-nia liniowego RL:

$$u = \left(\sqrt{x^2-1} + C\right)\sqrt{x^2-1}$$

$$u = x^2-1 + C\sqrt{x^2-1}$$

$$\underbrace{y^2 = x^2-1+C\sqrt{x^2-1}}_{} \quad \wedge \quad y(\sqrt{2}) = 0$$

$$0 = (\sqrt{2})^2 - 1 + C\sqrt{2-1}$$

$$0 = 1 + C \Rightarrow C = -1$$

R-nie szukanej krzywej całkowej:

$$y^2 = x^2-1-\sqrt{x^2-1}$$

R-nia rzędu II

(1) $y'' = 4\cos 2x$ z warunkiem początkowym: $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

całkujemy r-nie dwukrotnie:

$$y' = \int 4\cos 2x\,dx = 2\sin 2x + C$$

$$y = \int (2\sin 2x + C)\,dx = -\cos 2x + Cx + C_1 = y$$

$$y = -\cos 2x + Cx + C_1 \ \wedge\ y(0)=0$$

$$0 = -\underbrace{\cos 0}_{1} + \underbrace{C\cdot 0}_{0} + C_1 \Rightarrow C_1 = 1$$

$$y' = 2\sin 2x + C \ \wedge\ y'(0)=0$$

$$0 = 2\underbrace{\sin 0}_{0} + C \Rightarrow C = 0$$

$$y = -\cos 2x + 1$$

(2) $xy'' = y'\ln\frac{y'}{x}$ (nie występuje y)

Obniżamy rząd równania podstawiając: $y'' = u'$, $y' = u$

$$xu' = u\ln\frac{u}{x} \quad /:x$$

$$u' = \frac{u}{x} \ln \frac{u}{x}$$

$$\frac{du}{dx} = \frac{u}{x} \ln \frac{u}{x}$$

podstawienie $\frac{u}{x} = t \implies u = t \cdot x$ (t - f-ja x)

$$u' = t'x + t$$

$$t'x + t = t\ln t$$

$$t'x = t\ln t - t$$

$$\frac{dt}{dx} x = t(\ln t - 1) \quad /: dt$$

$$\frac{x}{dx} = \frac{t(\ln t - 1)}{dt}$$

$$\int \frac{dx}{x} = \int \frac{dt}{t(\ln t - 1)}$$

$$\int \frac{dt}{t(\ln t-1)} = \left\{ \begin{array}{l} \ln t - 1 = z \\ \frac{1}{t}\, dt = dz \end{array} \right\} = \int \frac{dz}{z} =$$

$$= \ln z = \ln|\ln t - 1| + C$$

$$\ln|x| = \ln(|\ln t - 1|) + C$$

$$x = C(\ln t - 1)$$

$$x = C\left(\ln \frac{u}{x} - 1\right)$$

$$x = C \ln \frac{u}{x} - C$$

$$x + C = C \ln \frac{u}{x} \quad /: C$$

$$\ln \frac{u}{x} = \frac{x}{C} + 1$$

$$\ln \frac{u}{x} = C x + 1 \qquad \left(C_1 = \frac{1}{C} \right)$$

$$\frac{u}{x} = e^{Cx + 1}$$

$$u = xe^{cx+1}$$

$$y' = xe^{cx+1}$$

$$dy = xe^{cx+1}\,dx$$

$$y = \int xe^{cx+1}dx = \int xe^{cx}e\,dx = e\int xe^{cx}dx$$

$$\int xe^{cx}dx = \begin{cases} f = x & g' = e^{cx} \\ f' = 1 & g = \frac{1}{c}e^{cx} \end{cases} = \frac{x}{c}e^{cx} - \frac{1}{c}\int e^{cx}dx = \frac{x}{c}e^{cx} - \frac{1}{c^2}e^{cx} + C_1$$

$$y = \frac{1}{c^2}e^{cx}(cx-1) + C_1$$

(3)

$$y''(x^2+1) = 2xy' \quad i \quad y(0)=1,\ y'(0)=3$$

$$\begin{matrix} y' = u \\ y'' = u' \end{matrix} \quad > \quad u'(x^2+1) = 2xu \quad /:(x^2+1)$$

$$\frac{du}{dx} = \frac{2xu}{x^2+1} \quad / \cdot \frac{dx}{u}$$

$$\frac{du}{u} = \frac{2x\,dx}{x^2+1}$$

$$\ln|u| = \ln(x^2+1) + C$$

$$u = C(x^2+1)$$

$$\begin{cases} y' = C(x^2+1) \\ \ldots \end{cases} \iff 3 = C(0^2+1)$$

$$\begin{cases} y' = C(x^2+1) \\ y'(0) = 3 \end{cases} \Leftrightarrow \quad 3 = C(0^2+1)$$

$$C = 3$$

$$y = \int (Cx^2 + C)\,dx \Rightarrow \quad y = \frac{C}{3}x^3 + Cx + C_1$$

$$\begin{cases} y = x^3 + 3x + C_1 \\ y(0) = 1 \end{cases}$$

$$\Updownarrow$$

$$1 = 0^3 + 3 \cdot 0 + C_1$$

$$C_1 = 1$$

$$y = x^3 + 3x + 1$$