

Matematyka 2

Wykład nr 2

1. Całkowanie funkcji trygonometrycznych

1. Całki funkcji trygonometrycznych możemy sprowadzić do całek funkcji wymiernych przy pomocy podstawienia uniwersalnego (tangens połówkowy):

podstawienie
uniwersalne :

$$t = \operatorname{tg} \frac{x}{2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

2. Jeśli funkcje $\sin x$ i $\cos x$ występują w parzystych potęgach, stosujemy podstawienie tangensowe:

$$t = \operatorname{tg} x$$

$$dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{t^2}{1+t^2}, \quad \cos^2 x = \frac{1}{1+t^2}.$$

Przykłady

$$\begin{aligned} 1. \int \frac{dx}{2+\cos x} &\rightarrow \text{podst. uniw.} \quad \left\{ \begin{array}{l} t = \operatorname{tg} \frac{x}{2} \\ dx = \frac{2}{1+t^2} dt \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} = \int \frac{\frac{2}{1+t^2} dt}{2 + \frac{1-t^2}{1+t^2}} = \\ &= 2 \int \frac{dt}{(1+t^2)[2 + \frac{1-t^2}{1+t^2}]} = 2 \int \frac{dt}{2+2t^2+1-t^2} = 2 \int \frac{dt}{t^2+3} = \\ &\quad \left\{ \begin{array}{l} t^2 = 3u^2 \\ t = \sqrt{3}u \\ dt = \sqrt{3}du \\ \rightarrow u = \frac{t}{\sqrt{3}} \end{array} \right\} = 2 \int \frac{\sqrt{3} du}{3u^2+3} = 2 \cdot \frac{\sqrt{3}}{3} \int \frac{du}{u^2+1} = 2 \frac{\sqrt{3}}{3} \operatorname{arctg} u + C = \\ &= \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{t}{\sqrt{3}} + C = \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{\operatorname{tg} \frac{x}{2}}{\sqrt{3}} + C \end{aligned}$$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$
 $a > 0$

$$\begin{aligned}
2. \int \frac{\sin x \cos x dx}{1 + \sin^4 x} &= \left\{ \begin{array}{l} t = \operatorname{tg} x \\ dx = \frac{dt}{1+t^2} \\ \sin^2 x = \frac{t^2}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2} \\ \sin x = \frac{t}{\sqrt{1+t^2}} \\ \cos x = \frac{1}{\sqrt{1+t^2}} \end{array} \right\} = \int \frac{\frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} dt}{1 + \left(\frac{t^2}{1+t^2}\right)^2} \frac{dt}{1+t^2} = \int \frac{\frac{t}{1+t^2} dt}{1+t^2 + \frac{t^4}{1+t^2}} = \\
&= \int \frac{t dt}{(1+t^2)^2 + t^4} = \int \frac{t dt}{2t^4 + 2t^2 + 1} = \\
&= \left\{ \begin{array}{l} t^2 = u \\ 2t dt = du \\ t dt = \frac{1}{2} du \end{array} \right\} = \frac{1}{2} \int \frac{du}{2u^2 + 2u + 1} = \\
&= \frac{1}{2} \int \frac{du}{2(u + \frac{1}{2})^2 + \frac{1}{2}} = \frac{1}{4} \int \frac{du}{(u + \frac{1}{2})^2 + \frac{1}{4}} = \left\{ \begin{array}{l} u + \frac{1}{2} = v \\ du = dv \end{array} \right\} = \frac{1}{4} \int \frac{dv}{v^2 + (\frac{1}{2})^2} = \\
&= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \cdot \operatorname{arctg} \frac{v}{\frac{1}{2}} = \frac{1}{2} \operatorname{arctg} 2(u + \frac{1}{2}) + C = \frac{1}{2} \operatorname{arctg} 2(t^2 + \frac{1}{2}) + C = \\
&= \frac{1}{2} \operatorname{arctg} (2\operatorname{tg}^2 x + 1) + C
\end{aligned}$$

Całki z ważniejszych funkcji trygonometrycznych

Wzór	Zakres zmienności
$\int \operatorname{tg} x dx = -\ln \cos x + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$
$\int \operatorname{ctg} x dx = \ln \sin x + C$	$x \neq k\pi, k \in \mathbb{Z}$
$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$	$x \in \mathbb{R}$
$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$	$x \in \mathbb{R}$
$\int \sin^3 x dx = -\cos x + \frac{\cos^3 x}{3} + C$	$x \in \mathbb{R}$
$\int \cos^3 x dx = \sin x - \frac{\sin^3 x}{3} + C$	$x \in \mathbb{R}$
$\int \frac{dx}{\sin x} = \ln \left \operatorname{tg} \frac{x}{2} \right + C$	$x \neq k\pi, k \in \mathbb{Z}$
$\int \frac{dx}{\cos x} = \ln \left \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$
$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2} \ln \left \operatorname{tg} \frac{x}{2} \right + C$	$x \neq k\pi, k \in \mathbb{Z}$
$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2} \ln \left \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

Wzory rekurencyjne:

$$\begin{aligned}
\int \sin^n x dx &= -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx, \quad n \geq 2, \\
\int \cos^n x dx &= \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx, \quad n \geq 2,
\end{aligned}$$

2. Całkowanie funkcji postaci $\sin ax \cos bx$, $\sin ax \sin bx$, $\cos ax \cos bx$.

Do obliczania powyższych całek stosujemy wzory trygonometryczne:

$$1^\circ \quad \sin ax \cos bx = \frac{1}{2} [\sin(a+b)x + \sin(a-b)x],$$

$$2^\circ \quad \sin ax \sin bx = \frac{1}{2} [\cos(a-b)x - \cos(a+b)x],$$

$$\cos ax \cos bx = \frac{1}{2} [\cos(a+b)x + \cos(a-b)x].$$

Przykłady

$$\begin{aligned} \int \sin 2x \cos 4x \, dx & \stackrel{1^\circ}{=} \frac{1}{2} \int (\sin 6x + \sin(-2x)) \, dx = \frac{1}{2} \int (\sin 6x - \sin 2x) \, dx = \\ &= \frac{1}{2} \int \sin 6x \, dx - \frac{1}{2} \int \sin 2x \, dx = \\ &= \frac{1}{2} \left(-\frac{1}{6} \cos 6x \right) - \frac{1}{2} \left(-\frac{1}{2} \cos 2x \right) + C \\ &= -\frac{1}{12} \cos 6x + \frac{1}{4} \cos 2x + C \end{aligned} \quad \left\{ \begin{array}{l} \int \sin ax \, dx = \\ = -\frac{1}{a} \cos ax \end{array} \right\}$$

$$\begin{aligned} \int \sin x \sin 3x \, dx & \stackrel{2^\circ}{=} \frac{1}{2} \int (\cos(-2x) - \cos 4x) \, dx = \frac{1}{2} \int \cos 2x \, dx - \frac{1}{2} \int \cos 4x \, dx = \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \sin 2x - \frac{1}{2} \cdot \frac{1}{4} \sin 4x + C = \frac{1}{4} \sin 2x - \frac{1}{8} \sin 4x + C \end{aligned}$$

$$\cos(-x) = \cos x \rightarrow \text{parzystość}$$

$$\sin(-x) = -\sin x \rightarrow \text{nieparzystość}$$

3. Całkowanie funkcji z niewymiernościami

Przydatne wzory:

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{x}{a} + c$$

$$\int \frac{1}{\sqrt{x^2 + k}} \, dx = \ln |x + \sqrt{x^2 + k}| + c$$

Przykłady

$$1) \int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \left\{ \begin{array}{l} t^6 = x \\ 6t^5 dt = dx \\ x^{\frac{1}{2}} = (t^6)^{\frac{1}{2}} = t^3 \\ x^{\frac{1}{3}} = t^2 \end{array} \right\} = \int \frac{6t^5 dt}{t^3 + t^2} = 6 \int \frac{t^3 dt}{t+1} = *$$

c. wymienna

Dzielimy: $t^3 : t+1 = t^2 - t + 1$

$$\begin{array}{r} t^3 : t+1 = t^2 - t + 1 \\ \underline{-t^3 - t^2} \\ -t^2 + t \\ \underline{+t^2 + t} \\ t \\ \underline{-t - 1} \\ -1 \end{array}$$

$$* = 6 \int (t^2 - t + 1 - \frac{1}{t+1}) dt = 6 \left[\frac{1}{3}t^3 - \frac{1}{2}t^2 + t - \ln|t+1| \right] + C =$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + \sqrt[6]{x} - \ln|\sqrt[6]{x}| + C$$

$$2) \int \sqrt[3]{\frac{x+1}{x-1}} \frac{dx}{x+1} = \left\{ \begin{array}{l} \sqrt[3]{\frac{x+1}{x-1}} = t \quad | (*)^3 \\ \frac{x+1}{x-1} = t^3 \quad | (x-1) \\ x+1 = t^3 x - t^3 \\ x(1-t^3) = -t^3 - 1 \\ x = \frac{-(t^3+1)}{1-t^3} \\ \boxed{x = \frac{t^3+1}{t^3-1}} \\ dx = \frac{3t^2(t^3-1) - (t^3+1)3t^2}{(t^3-1)^2} dt \end{array} \right.$$

$$dx = \frac{-3t^2 - 3t^2}{(t^3-1)^2} dt$$

$$dx = \frac{-6t^2}{(t^3-1)^2} dt$$

$$= \int t \cdot \frac{\frac{-6t^2}{(t^3-1)^2} dt}{\frac{t^3+1}{t^3-1} + 1} = \int -\frac{6t^2}{(t^3-1)^2} \cdot \frac{1}{\frac{2t^3}{t^3-1}} dt = -3 \int \frac{1}{t^3-1} dt$$

y

$$y = \int \frac{1}{(t-1)(t^2+t+1)} dt =$$

$$= \frac{1}{3} \int \frac{1}{t-1} dt + \int \frac{-\frac{1}{3}t - \frac{2}{3}}{t^2+t+1} dt =$$

$$= \frac{1}{3} \int \frac{1}{t-1} dt + \frac{1}{3} \int \frac{t+2}{t^2+t+1} dt =$$

$$= \frac{1}{3} \ln|t-1| - \frac{1}{3} y_1$$

$$y_1 = \int \frac{\frac{1}{2}(2t+1) + \frac{3}{2}}{t^2+t+1} dt =$$

$$= \frac{1}{2} \int \frac{2t+1}{t^2+t+1} dt + \frac{3}{2} \int \frac{1}{t^2+t+1} dt = \frac{1}{2} \ln(t^2+t+1) + \frac{3}{2} y_2$$

y_2

$$y_2 = \int \frac{1}{(t+\frac{1}{2})^2 + \frac{3}{4}} dt = \int \frac{1}{(t+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dt = \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arctg} \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{2t+1}{\sqrt{3}} + C$$

$$\frac{1}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \quad | \cdot (t-1)(t^2+t+1)$$

$$1 = \underbrace{A}_{t^2} + \underbrace{A}_{t^1} + \underbrace{A}_{t^0} + \underbrace{Bt^2}_{t^2} + \underbrace{Ct}_{t^1} + \underbrace{-Bt}_{t^1} + \underbrace{-C}_{t^0}$$

$$t^2: 0 = A+B \Rightarrow A = -B$$

$$t: 0 = A+C-B \quad \left\{ \begin{array}{l} C-2B=0 \\ -B-C=1 \end{array} \right.$$

$$w: 1 = A-C \quad + \quad -B-C=1$$

$$C = A+1$$

$$-3B=1 \Rightarrow B = -\frac{1}{3}$$

$$A = \frac{1}{3}$$

$$\left\{ \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C \right.$$

3)

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2+2x+4}} dx &= \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+4}} dx = \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+4}} dx + \int \frac{1}{\sqrt{x^2+2x+4}} dx = \\ &= \sqrt{x^2+2x+4} + \underbrace{\int \frac{1}{\sqrt{(x+1)^2+3}} dx}_{\text{wzór}} \end{aligned}$$

$$\begin{aligned} y &= \left\{ \begin{array}{l} x+1=t \\ dx=dt \end{array} \right\} = \int \frac{1}{\sqrt{t^2+3}} dt \stackrel{\text{wzór}}{=} \ln|t + \sqrt{t^2+3}| + C = \\ &= \ln|x+1 + \sqrt{x^2+2x+4}| + C \end{aligned}$$

4)

$$\int \frac{1-x}{\sqrt{4+2x-x^2}} dx = \int \frac{\frac{1}{2}(-2x+2)}{\sqrt{4+2x-x^2}} dx = \frac{1}{2} \cdot \int \frac{-2x+2}{\sqrt{4+2x-x^2}} dx = \sqrt{4+2x-x^2} + C$$

5)

$$\int \frac{dx}{\sqrt{4+2x-x^2}} = \int \frac{dx}{\sqrt{5-(x-1)^2}} = \left\{ \begin{array}{l} x-1=t \\ dx=dt \end{array} \right\} = \int \frac{dt}{\sqrt{5-t^2}} = \int \frac{dt}{\sqrt{(\sqrt{5})^2-t^2}} =$$

$$\stackrel{\text{wzór}}{=} \arcsin \frac{t}{\sqrt{5}} + C = \arcsin \frac{x-1}{\sqrt{5}} + C$$

Twierdzenie 3.1 (metoda współczynników nieoznaczonych)

Do obliczenia całki postaci:

$$\int \frac{W_n(x)}{\sqrt{ax^2+bx+c}} dx,$$

stosujemy wzór:

$$\int \frac{W_n(x)}{\sqrt{ax^2+bx+c}} dx = Q_{n-1}(x) \sqrt{ax^2+bx+c} + \int \frac{A}{\sqrt{ax^2+bx+c}} dx, \text{ gdzie } a \neq 0.$$

$W_n(x)$ – dany wielomian stopnia n ;

$Q_{n-1}(x)$ – szukany wielomian stopnia $n-1$ o nieznanych współczynnikach;

A – szukana liczba rzeczywista.

Przykład

$$W_n = x^2 + 4x + 1 \Rightarrow Q_{n-1} = mx + n, \quad m, n \in \mathbb{R} \quad A \in \mathbb{R}$$

$$\int \frac{x^2 + 4x + 1}{\sqrt{x^2 + 2x + 6}} dx = (mx + n) \cdot \sqrt{x^2 + 2x + 6} + \int \frac{A}{\sqrt{x^2 + 2x + 6}} dx$$

Różniczkując obie strony równania

$$\frac{x^2 + 4x + 1}{\sqrt{x^2 + 2x + 6}} = m \cdot \sqrt{x^2 + 2x + 6} + (mx + n) \cdot \frac{2x + 2}{2\sqrt{x^2 + 2x + 6}} + \frac{A}{\sqrt{x^2 + 2x + 6}} \cdot \frac{1}{\sqrt{x^2 + 2x + 6}}$$

$$x^2 + 4x + 1 = m(x^2 + 2x + 6) + (mx + n)(x + 1) + A$$

$$x^2 + 4x + 1 = \underline{mx^2} + \underline{2mx} + 6m + \underline{mx^2} + \underline{mx} + \underline{nx} + n + A$$

$$x^2: \quad 1 = 2m \Rightarrow m = \frac{1}{2}$$

$$x: \quad 4 = 3m + n \Rightarrow n = \frac{5}{2}$$

$$w: \quad 1 = 6m + n + A \Rightarrow A = 1 - \frac{11}{2} = -\frac{9}{2}$$

$$\rightarrow \int \frac{x^2 + 4x + 1}{\sqrt{x^2 + 2x + 6}} dx = \left(\frac{1}{2}x + \frac{5}{2}\right) \sqrt{x^2 + 2x + 6} - \frac{9}{2} \int \frac{1}{\sqrt{x^2 + 2x + 6}} dx$$

$$\begin{aligned} \int \frac{1}{\sqrt{(x+1)^2 + 5}} dx &= \\ &= \ln |x+1 + \sqrt{x^2 + 2x + 6}| + C \end{aligned}$$

4. Podstawienia Eulera

Twierdzenie 4.1

Całkę z funkcji zawierającej x oraz $\sqrt{ax^2 + bx + c}$,

gdzie $a \neq 0$ i $\Delta = b^2 - 4ac \neq 0$, sprowadzamy do całki z funkcji wymiernej stosując podstawienia:

I. (pierwsze podstawienie Eulera)

$$\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}, \quad \text{jeśli } a > 0;$$

II. (drugie podstawienie Eulera)

$$\sqrt{ax^2 + bx + c} = xt - \sqrt{c}, \quad \text{jeśli } c > 0;$$

III. (trzecie podstawienie Eulera)

$$\sqrt{ax^2 + bx + c} = \sqrt{a(x - x_1)(x - x_2)} = x(x - x_1)t, \quad \text{jeśli } \Delta > 0.$$

W podstawieniu II, różnicę $xt - \sqrt{c}$ można zastąpić sumą $xt + \sqrt{c}$, lub różnicą $\sqrt{c} - xt$.

Stosując I podstawienie Eulera obliczymy całkę

$$\int \frac{dx}{x\sqrt{x^2-2x}} = \left\{ \begin{array}{l} \sqrt{x^2-2x} = t-x \quad |(\)^2 \Rightarrow t = x + \sqrt{x^2-2x} \\ \cancel{x^2} - 2x = t^2 - 2tx + \cancel{x^2} \\ -2x + 2tx = t^2 \\ 2x(t-1) = t^2 \\ 2x = \frac{t^2}{t-1} \Rightarrow x = \frac{t^2}{2(t-1)} \end{array} \right.$$

$$dx = \frac{4t(t-1) - t^2 \cdot 2}{4(t-1)^2} dt$$

$$dx = \frac{2t^2 - 4t}{4(t-1)^2} dt$$

$$dx = \frac{t^2 - 2t}{2(t-1)^2} dt$$

$$= \int \frac{\frac{t^2-2t}{2(t-1)^2} dt}{\frac{t^2}{2(t-1)} \cdot (t - \frac{t^2}{2(t-1)})} = \int \frac{t^2-2t}{2(t-1)^2} \cdot \frac{1}{\frac{t^3}{2(t-1)} - \frac{t^4}{4(t-1)^2}} dt =$$

$$= \int \frac{t^2-2t}{2(t-1)^2 \cdot \frac{2t^3(t-1) - t^4}{4(t-1)^2}} dt = \int \frac{t^2-2t}{2t^4 - 2t^3 - t^4} dt = \int \frac{t^2-2t}{t^4-2t^3} dt =$$

$$= \int \frac{t(t-2)}{t^3(t-2)} dt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C = -\frac{1}{x + \sqrt{x^2-2x}} + C$$

II sposób

$$\int \frac{dx}{x\sqrt{x^2-2x}} = \int \frac{dx}{x\sqrt{x^2(1-\frac{2}{x})}} = \int \frac{dx}{x^2\sqrt{1-\frac{2}{x}}} = \left\{ \begin{array}{l} 1 - \frac{2}{x} = t \\ \frac{2}{x^2} dx = dt \\ \frac{dx}{x^2} = \frac{1}{2} dt \end{array} \right.$$

$$= \int \frac{\frac{1}{2} dt}{\sqrt{t}} = \frac{1}{2} \int t^{-\frac{1}{2}} dt = \frac{1}{2} \cdot 2\sqrt{t} + C = \sqrt{1-\frac{2}{x}} + C$$

$\int \frac{x^2}{\sqrt{x^2+1}} dx$ \rightarrow m. współczynnika nieoznaczonego

$$= \int x \cdot \frac{x}{\sqrt{x^2+1}} dx = \left\{ \begin{array}{l} f = x \quad g' = \frac{1}{2} \frac{2x}{\sqrt{x^2+1}} \\ f' = 1 \quad g = \sqrt{x^2+1} \end{array} \right\} = \left\{ \int \frac{f'}{\sqrt{f}} = 2\sqrt{f} \right\}$$

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1} dx = x\sqrt{x^2+1} - \int \frac{x^2+1}{\sqrt{x^2+1}} dx =$$

$$= x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}} dx - \int \frac{1}{\sqrt{x^2+1}} dx$$

$$\int \frac{x^2}{\sqrt{x^2+1}} dx = x\sqrt{x^2+1} - \int \frac{x^2}{\sqrt{x^2+1}} dx - \ln|x+\sqrt{x^2+1}|$$

$$2 \int \frac{x^2}{\sqrt{x^2+1}} dx = x\sqrt{x^2+1} - \ln|x+\sqrt{x^2+1}| \quad | : 2$$

⋮

+ C