Ćwiczenia nr 11

poniedziałek, 31 maja 2021

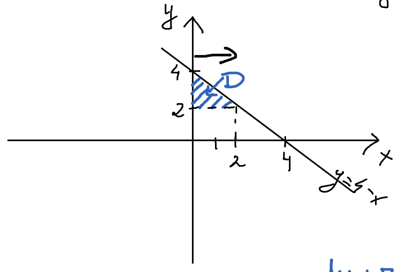
Zad. 1

W całkach iterowanych zamienić kolejność całkowania

a) $\int_0^2 \left(\int_2^{4-x} f(x,y)\,dy \right) dx$

$y = 4-x$

$D: \begin{cases} 0 \le x \le 2 \\ 2 \le y \le 4-x \end{cases}$



$\int_2^4 \left(\int_0^{4-y} f(x,y)\,dx \right) dy$

b) $\int_0^2 \left(\int_{2x}^{6-x} f(x,y)\,dy \right) dx$

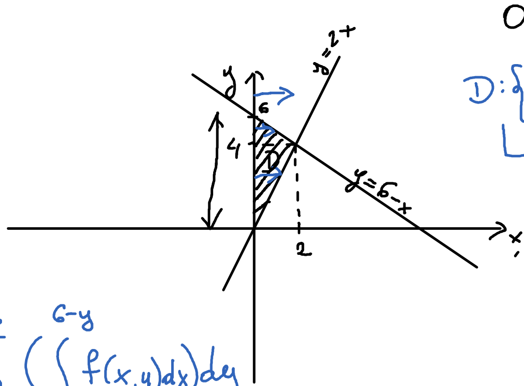
$D: \begin{cases} 0 \le x \le 2 \\ 2x \le y \le 6-x \end{cases}$ normalny wzgl. OX

$0 \le x$

$0 \le y \le 6$

$D: \begin{cases} 0 \le x \le \frac{y}{2} \\ 0 \le y \le 4 \end{cases} \vee \begin{cases} 0 \le x \le 6-y \\ 4 \le y \le 6 \end{cases}$

normalny wzgl OY



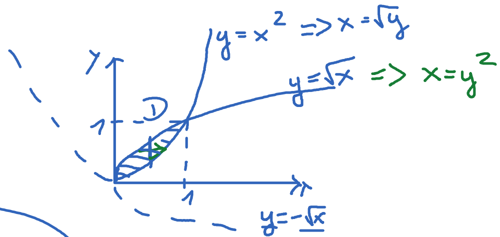
$\int_0^4 \left(\int_0^{\frac{y}{2}} f(x,y)\,dx \right) dy + \int_4^6 \left(\int_0^{6-y} f(x,y)\,dx \right) dy$

Zad. 2.

Zdefiniować obszar D ograniczony podanymi krzywymi jako normalny względem osi OX, osi OY (na dwa sposoby).

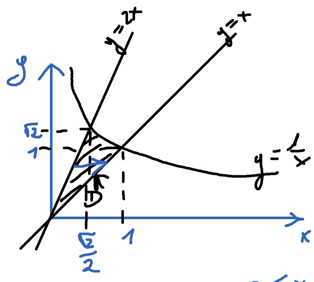
a) $y = x^2$, $y = \sqrt{x}$

$y^2 = x$



$D: \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{cases}$ lub $D: \begin{cases} y^2 \le x \le \sqrt{y} \\ 0 \le y \le 1 \end{cases}$

b) $y = \frac{1}{x}$, $y = x$, $y = 2x$ (dla $x>0$)



Obliczyć pole obszaru D

$|D| = \iint_D dx\,dy$

$D: \begin{cases} 0 \le x \le \frac{\sqrt{2}}{2} \\ x \le y \le 2x \end{cases} \cup \begin{cases} \frac{\sqrt{2}}{2} \le x \le 1 \\ x \le y \le \frac{1}{x} \end{cases}$

lub

$D: \begin{cases} \frac{y}{2} \le x \le y \\ 0 \le y \le 1 \end{cases} \cup \begin{cases} \frac{y}{2} \le x \le \frac{1}{y} \\ 1 \le y \le \sqrt{2} \end{cases}$

$|D| = \iint\limits_D dx\,dy$

lub

$$D: \begin{cases} \frac{y}{2} \le x \le y \\ 0 \le y \le 1 \end{cases} \cup \begin{cases} \frac{y}{2} \le x \le \frac{1}{y} \\ 1 \le y \le \sqrt{2} \end{cases}$$

$$|D| = \int_0^1 \left(\int_{\frac{y}{2}}^{y} dx \right) dy + \int_1^{\sqrt{2}} \left(\int_{\frac{y}{2}}^{\frac{1}{y}} dx \right) dy =$$

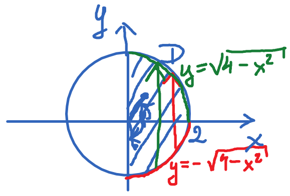
$$= \int_0^1 \left[x \Big|_{\frac{y}{2}}^{y} \right] dy + \int_1^{\sqrt{2}} \left[x \Big|_{\frac{y}{2}}^{\frac{1}{y}} \right] dy = \int_0^1 \left(y - \frac{y}{2} \right) dy + \int_1^{\sqrt{2}} \left(\frac{1}{y} - \frac{y}{2} \right) dy =$$

$$= \int_0^1 \frac{y}{2} dy + \int_1^{\sqrt{2}} \left(\frac{1}{y} - \frac{y}{2} \right) dy = \frac{1}{4} y^2 \Big|_0^1 + \left[\ln|y| - \frac{1}{4} y^2 \right] \Big|_1^{\sqrt{2}} =$$

$$= \frac{1}{4}(1 - 0) + \left[\left(\ln\sqrt{2} - \frac{1}{2} \right) - \left(\underbrace{\ln 1}_{0} - \frac{1}{4} \right) \right] = \frac{1}{4} + \ln\sqrt{2} - \frac{1}{2} + \frac{1}{4} = \underline{\underline{\ln\sqrt{2}}}$$

Zad. 3.

Obliczyć $\iint\limits_D x y^2 \, dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 4 \land x \ge 0\}$



współrzędne biegunowe

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\boxed{\iint\limits_D f(x,y)\,dx = \iint f(r\cos\varphi, r\sin\varphi) \cdot r \, dr\,d\varphi}$$

$$\begin{cases} 0 \le r \le 2 \\ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$D: \begin{cases} 0 \le x \le \sqrt{4-y^2} \\ -2 \le y \le 2 \end{cases}$$

lub

$$D. \begin{cases} 0 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{cases}$$

$$\int_0^2 \left(x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2 \, dy \right) dx = \dots$$

$$\iint\limits_D x y^2 \, dx\,dy = \int_0^2 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^3 \cos\varphi \sin^2\varphi \right) r\,dr =$$

$$= \int_0^2 r^4 \, dr \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi \, d\varphi =$$

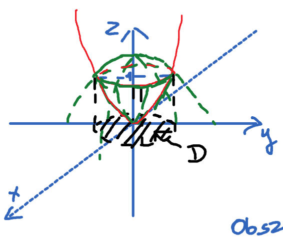
$$= \frac{1}{5} r^5 \Big|_0^2 \cdot \frac{1}{3} \sin^3\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{32}{5} \cdot \frac{1}{3}(1+1) = \frac{32}{5} \cdot \frac{2}{3} = \underline{\underline{\frac{64}{15}}}$$

Zad. 4.

Obliczyć objętość bryły ograniczonej paraboloidą $2z = x^2 + y^2$ i sferą $x^2 + y^2 + z^2 = 3$ $\quad z = \sqrt{3 - x^2 - y^2}$

(część wewnątrz paraboloidy)

$$V = \iint\limits_D \left(\sqrt{3 - x^2 - y^2} - \frac{x^2 + y^2}{2} \right) dx\,dy$$



Obszar D: $\quad z > 0$

$$\begin{cases} 2z = x^2 + y^2 \\ \underbrace{x^2 + y^2}_{2z} + z^2 = 3 \end{cases} \Rightarrow 2z + z^2 = 3$$

$$\underset{\sim}{z = 1}$$

$D: \; x^2 + y^2 \le 2$

wsp. biegunowe

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases},$$

$$\begin{cases} 0 \le r \le \sqrt{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\begin{cases} 0 \le r \le \sqrt{2} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$V = \int_0^{\sqrt{2}} \left(\int_0^{2\pi} \left(\sqrt{3-r^2} - \frac{r^2}{2}\right) d\varphi \right) r\, dr \quad \int_0^{\sqrt{2}} \left[\left(\sqrt{3-r^2} - \frac{r^2}{2}\right) r \int_0^{2\pi} d\varphi \right] dr =$$

$$= \int_0^{\sqrt{2}} \left(\sqrt{3-r^2} - \frac{r^2}{2}\right) r\, \varphi \Big|_0^{2\pi} dr = 2\pi \int_0^{\sqrt{2}} \left(\sqrt{3-r^2}\, r - \frac{r^3}{2}\right) dr =$$

$$= 2\pi \left[\int_0^{\sqrt{2}} r\sqrt{3-r^2}\, dr - \left(\frac{1}{8} r^4\right)\Big|_0^{\sqrt{2}} \right] = \ldots$$

Zad. 5.

Znaleźć objętość bryły ograniczonej powierzchniami:

$$z = 1 + x + y \; ; \; x = 0 \; ; \; y = 0 \; ; \; z = 0 \; ; \; x + y = 1$$

$$\iint_D (1 + x + y)\, dx\, dy \, ,$$

D: