Ćwiczenia nr 10

poniedziałek, 24 maja 2021

1 Wyznaczyć ekstrema lokalne podanych f-cji:

a) $f(x,y) = 2y^3x + x^2 + 3y^2$

b) $f(x,y) = x^2y - xy^2 + \frac{1}{3}y^3 - 9y$

c) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

2. Znaleźć wymiary prostopadłościennej wanny która przy danej objętości V ma najmniejszą powierzchnię

Rozwiązania

1 a) $f(x,y) = 2y^3x + x^2 + 3y^2 \qquad D_f = \mathbb{R}^2$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2y^3 + 2x = 0 \\ 6xy^2 + 6y = 0 \end{cases} \Leftrightarrow \begin{cases} x + y^3 = 0 \\ y(xy+1) = 0 \end{cases} \Rightarrow \text{p-ty stacjonarne}$$

$P_1(0,0)$, $P_2(-1,1)$, $P_3(1,-1)$

obliczamy pochodne rzędu drugiego:

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial(2y^3 + 2x)}{\partial x} = 2$$

$$f''_{yy} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial(6xy^2 + 6y)}{\partial y} = 12xy + 6$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial(2y^3 + 2x)}{\partial y} = 6y^2$$

$$W = \begin{vmatrix} 2 & 6y^2 \\ 6y^2 & 12xy + 6 \end{vmatrix}$$

Mamy:

$$W(P_1) = W(\underset{x,y}{(0,0)}) = \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} = 12 > 0 \Rightarrow \text{w } P_1 \text{ jest ekstremum}$$

ponieważ $\frac{\partial^2 f}{\partial x^2} > 0$, to minimum lokalne

$\left(\frac{\partial^2 f}{\partial y^2} > 0\right)$

$$W(P_2) = W(\overset{x\ y}{(-1,1)}) = \begin{vmatrix} 2 & 6 \\ 6 & -6 \end{vmatrix} = -12 - 36 < 0, \text{ to w } P_2 \text{ nie ma ekstremum}$$

$$W(P_3) = W((1,-1)) = \begin{vmatrix} 2 & 6 \\ 6 & -6 \end{vmatrix} = -12 - 36 < 0, \text{ to w } P_3 \text{ nie ma ekstrem}$$

Odp: $f_{min}((0,0)) = 0$

1b) $f(x,y) = x^2y - xy^2 + \frac{1}{3}y^3 - 9y \qquad D = \mathbb{R}^2$

1c) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f'_x = 4x^3 - 4x + 4y$

$f'_y = 4y^3 + 4x - 4y$

WK:

$$1^\circ \begin{cases} x^3 - x + y = 0 \\ y^3 + x - y = 0 \end{cases}$$

$f'_x = 4x^3 - 4x + 4y$

$f'_y = 4y^3 + 4x - 4y$

wk.

$$1^\circ \begin{cases} x^3 - x + y = 0 \\ y^3 + x - y = 0 \end{cases} \quad +$$

$$x^3 + y^3 = 0$$

$$y = -x$$

$$1^\circ \quad x^3 - x - x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2-2) = 0$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = -\sqrt{2} \\ y = \sqrt{2} \end{cases} \vee \begin{cases} x = \sqrt{2} \\ y = -\sqrt{2} \end{cases}$$

$f''_{xx} = 12x^2 - 4$

$f''_{yy} = 12y^2 - 4$

$f''_{xy} = 4$

$$W = \begin{vmatrix} 12x^2-4 & 4 \\ 4 & 12y^2-4 \end{vmatrix}$$

$$W((0,0)) = \begin{vmatrix} -4 & 4 \\ 4 & -4 \end{vmatrix} = 16 - 16 = 0 \Rightarrow ?$$

$$W((-\sqrt{2}, \sqrt{2})) = \begin{vmatrix} 20 & 4 \\ 4 & 20 \end{vmatrix} > 0 \Rightarrow f_{min}(-\sqrt{2}, \sqrt{2}) = \ldots$$

$$W((\sqrt{2}, -\sqrt{2})) = \begin{vmatrix} 20 & 4 \\ 4 & 20 \end{vmatrix} \Rightarrow \text{j.w} \quad f_{min}(\sqrt{2}, -\sqrt{2}) = \ldots$$

Dla p-tu $(0,0)$ sprawdzamy, czy istnieje $\underline{f_{min}(\overset{x}{0}, \overset{y}{0}) = 0}$ lub ew. $f_{max}(0,0) = 0$

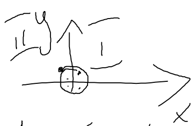
dla wszystkich (x,y) „blisko" $(0,0)$ $f(x,y) > 0$

analogicznie dla f_{max}:

$f(x,y) < 0$ dla wszystkich (x,y) „blisko" $(0,0)$

$n \in \mathbb{N}$

Obliczmy $f(\overset{x}{\tfrac{1}{n}}, \overset{y}{\tfrac{1}{n}}) = \frac{1}{n^4} + \frac{1}{n^4} - \frac{2}{n^2} + \frac{4}{n^2} - \frac{2}{n^2} = \frac{2}{n^4} > 0$

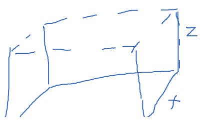


$$f\left(-\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n^4} + \frac{1}{n^4} - \frac{2}{n^2} - \frac{4}{n^2} - \frac{2}{n^2} = \frac{2}{n^4} - \frac{8}{n^2} = \overset{>0}{\frac{2}{n^2}} \overbrace{\left(\frac{1}{n^2} - 4\right)}^{\leq 0} < 0$$

Dowolnie blisko p-tu $(0,0)$ f-cja przyjmuje zarówno wartości dodatnie jak i ujemne.
Tak więc brak ekstremum w p-cie $(0,0)$

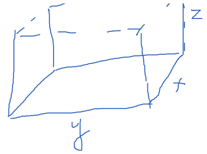
2. Znaleźć wymiary prostopadłościennej wanny, która przy danej objętości V ma najmniejszą powierzchnię

$x > 0$



$$V = xyz \Rightarrow z = \frac{V}{xy}$$

... V ma najmniejszą powe...



$$V = xyz \Rightarrow z = \frac{V}{xy}$$

$$P_p = xy + 2xz + 2yz$$

$$P(x,y) = xy + \frac{2V}{y} + \frac{2V}{x}$$

$x>0$
$y>0$
$z>0$

$$P(x,y) = xy + \frac{2V}{y} + \frac{2V}{x}$$

Szukamy minimum lok.

$$P'_x = y - \frac{2V}{x^2}$$

$$P'_y = x - \frac{2V}{y^2}$$

$$\begin{cases} y = \frac{2V}{x^2} \\ x - \frac{\cancel{2V} \cdot x^4}{(2V)^2} = 0 \end{cases} \iff x - \frac{x^4}{2V} = 0$$

$$x\left(1 - \frac{x^3}{2V}\right) = 0$$

$$x = 0 \notin D \qquad x = \sqrt[3]{2V} \Rightarrow y = \frac{2V}{(2V)^{\frac{2}{3}}} = \sqrt[3]{2V}$$

$$\left(\sqrt[3]{2V}, \sqrt[3]{2V}\right)$$

$$P''_{xx} = -2V \cdot (-2x^{-3}) = 4\frac{V}{x^3}$$

$$P''_{yy} = 4\frac{V}{y^3}$$

$$P''_{xy} = P''_{yx} = 1$$

$$W = \begin{vmatrix} \frac{4V}{x^3} & 1 \\ 1 & \frac{4V}{y^3} \end{vmatrix}$$

$$W\left(\left(\sqrt[3]{2V}, \sqrt[3]{2V}\right)\right) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$$

$$\Rightarrow P_{min}\left(\sqrt[3]{2V}, \sqrt[3]{2V}\right) = \dots$$

$$z = \frac{V}{\sqrt[3]{2V} \cdot \sqrt[3]{2V}} = \frac{V}{\sqrt[3]{4V^2}} = \frac{1}{\sqrt[3]{4}} V^{\frac{1}{3}} = \frac{1}{\sqrt[3]{4V}}$$

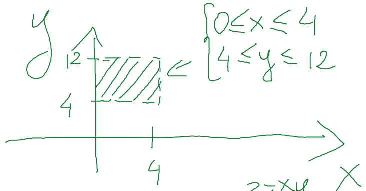
$$y = x = \sqrt[3]{2V} \quad \text{i} \quad z = \frac{1}{\sqrt[3]{4V}}$$

Całki podwójne

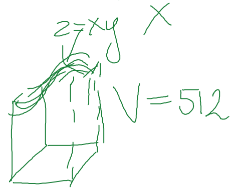
Obliczyć

(1) $\int_0^4 dx \int_4^{12} xy\,dy$

$f(x,y) = xy \leftarrow$ powierzchnia w $\mathbb{R}^3$



$\begin{cases} 0 \le x \le 4 \\ 4 \le y \le 12 \end{cases}$



(2) $\underbrace{\int_0^1 dx \int_0^2 xy(x-y)\,dy}$ $= \int_0^1 \Big(\underbrace{\int_0^2 xy(x-y)\,dy}_{J} \Big) dx = (*)$

$f(x,y) = x^2y - xy^2$

$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$

$$J = \int_0^2 (x^2y - xy^2)\,dy = x^2 \int_0^2 y\,dy - x \int_0^2 y^2\,dy = x^2 \cdot \left[\tfrac{1}{2}y^2\right]\Big|_0^2 - x\left[\tfrac{1}{3}y^3\right]_0^2 =$$

$$= x^2(2-0) - x\left(\tfrac{8}{3} - 0\right) = 2x^2 - \tfrac{8}{3}x$$

$$(*) = \int_0^1 \left(2x^3 - \tfrac{8}{3}x\right)dx = \left[\tfrac{1}{2}x^4 - \tfrac{4}{3}x^2\right]\Big|_0^1 = \tfrac{1}{2} - \tfrac{4}{3} - 0 = \tfrac{3-8}{6} = \underline{\underline{-\tfrac{5}{6}}}$$

(1) Uwaga!

Czasami całka podwójna może być interpretowana jako iloczyn c. pojed.

$$\int_0^4 \Big(\int_4^{12} \underbrace{xy}_{x \cdot y}\,dy \Big) dx = \int_0^4 \Big(x \int_4^{12} y\,dy \Big) dx = \int_0^4 \Big(x \left[\tfrac{1}{2}y^2\right]\Big|_4^{12} \Big) dx =$$

$$= \int^4 \left(x \left[\tfrac{1}{2} \cdot 144 - \tfrac{1}{2} \cdot 16 \right] \right) \ldots$$

$$= \int_0^4 \left(x \left[\tfrac{1}{2} \cdot 144 - \tfrac{1}{2} \cdot 16 \right] \right) dx = \int_0^4 64x\,dx = 64 \int_0^4 x\,dx = 64 \cdot \tfrac{1}{2} x^2 \Big|_0^4 = 32[16-0] = \underline{\underline{512}}$$

$x \cdot y$, $72 - 8$

$$\int_0^4 x\,dx \cdot \int_4^{12} y\,dy = \tfrac{1}{2}x^2 \Big|_0^4 \cdot \tfrac{1}{2} y^2 \Big|_4^{12} = \tfrac{1}{2}(16-0) \cdot \tfrac{1}{2}(144-16) = \tfrac{1}{4} \cdot 16 \cdot 128 = \underline{\underline{512}}$$

!! $\boxed{f(x) \cdot f(y)}$!!

Uwaga!

$f(x,y)=1$

$$\boxed{\iint_D dx\,dy = \text{pole obszaru } D}$$

$V = P_D \cdot 1 = P_D$

(3) Oblicz $\iint_D (3xy^2 - x^2y)\,dx\,dy$, gdzie D jest prostokątem $D: \begin{cases} -1 \le x \le 1 \\ 0 \le y \le 2 \end{cases}$

$$\int_{-1}^{1} \left(\int_0^2 (3xy^2 - x^2y)\,dy \right) dx = \int_0^2 \left(\int_{-1}^{1} (3xy^2 - x^2y)\,dx \right) dy$$

Gdy obszar całkowania jest prostokątem, kolejność całkowania nie ma znaczenia

$$\int_{-1}^{1} \left(\left[xy^3 - \tfrac{1}{2}x^2y^2 \right] \Big|_0^2 \right) dx = \int_{-1}^{1} \left(x \cdot 8 - \tfrac{1}{2}x^2 \cdot 4 - 0 + 0 \right) dx = \int_{-1}^{1} (8x - 2x^2)\,dx = \left(4x^2 - \tfrac{2}{3}x^3\right) \Big|_{-1}^{1} =$$

$$= 4 - \tfrac{2}{3} - \left(4 + \tfrac{2}{3}\right) = \underline{-\tfrac{4}{3}}$$

$\int_0^2 ([2\ 2\ 2 \quad 3\ 7)\ 1)\ ,\ \ 2\ (\ [\quad 2 \quad 2$

$$\int_0^2 \left(\left[\frac{2}{3}x^2y^2 - \frac{1}{3}x^3y\right]\Big|_{-1}^{1} \right) dy = \int_0^2 \left[\frac{2}{3}y^2 - \frac{1}{3}y - \left(\frac{2}{3}y^2 + \frac{1}{3}y\right)\right] dy = \int_0^2 -\frac{2}{3}y\, dy = -\frac{1}{3}y^2\Big|_0^2 = -\frac{4}{3} + 0 = -\frac{4}{3}$$

(4) $\iint_D e^{x+y}\, dx\,dy$, $D = \langle 0,1\rangle \times \langle 0,1\rangle$

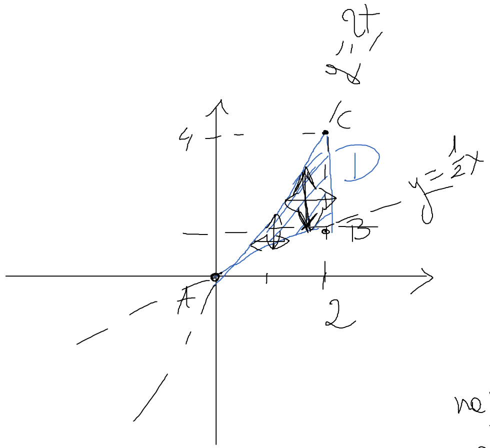
$D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$ $\qquad e^{x+y} = e^x \cdot e^y$

$$\int_0^1 \left(\int_0^1 e^{x+y} dy \right) dx = \int_0^1 \left(\int_0^1 e^x \cdot e^y dy \right) dx = \int_0^1 e^x dx \cdot \int_0^1 e^y dy = e^x\Big|_0^1 \cdot e^y\Big|_0^1 = (e^1 - e^0)(e^1 - e^0) = (e-1)^2$$

Zad. 2.

Obliczyć $\iint_D \frac{1}{(1+x+y)^2} dx\,dy$, gdzie $D = \triangle ABC$,

$A=(0,0)$, $B=(2,1)$, $C=(2,4)$



Opisuję obszar D

$D: \begin{cases} 0 \le x \le 2 \\ \frac{1}{2}x \le y \le 2x \end{cases}$ lub $\cancel{\begin{cases} \le x \le \\ 0 \le y \le 2 \end{cases}}$ plus $\cancel{\begin{cases} \le x \le \\ 2 \le y \le 4 \end{cases}}$

najpierw obliczamy całkę po y (na końcu po zmiennej, która przebiega przedział liczbowy)

$$\int_0^2 \left(\int_{\frac{1}{2}x}^{2x} \frac{1}{(1+x+y)^2} dy \right) dx$$

$u = 2x$

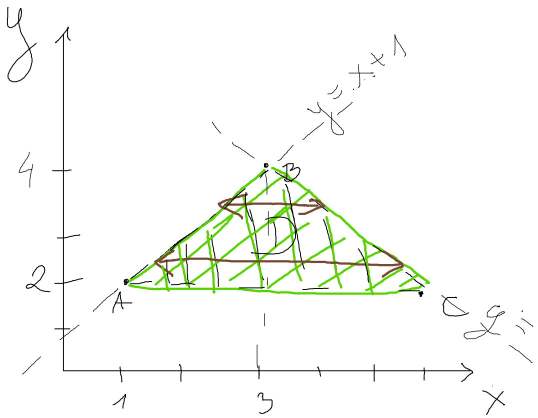
$$\underbrace{\int_0 \Big(\int_{\frac{1}{2}x} (1+x+y) \quad \Big) dx}$$

$$y = -\frac{1}{1+x+y}\Bigg|_{y=\frac{1}{2}x}^{y=2x} = -\frac{1}{1+x+2x} - \left(-\frac{1}{1+x+\frac{1}{2}x}\right) = -\frac{1}{1+3x} + \frac{1}{1+\frac{3}{2}x}$$

$$\int_0^2 \left(\frac{1}{1+\frac{3}{2}x} - \frac{1}{1+3x}\right)dx = \left(\frac{2}{3}\ln\left|1+\frac{3}{2}x\right| - \frac{1}{3}\ln|1+3x|\right)\Bigg|_0^2 = \frac{2}{3}\ln 4 - \frac{1}{3}\ln 7$$

Zad. 3.

$\iint_D (x+1)\,dx\,dy$, gdzie D jest trójkątem o wierzchołkach $A=(1,2)$, $B=(3,4)$, $C=(6,2)$



$$D: \begin{cases} 0 \le x \le 3 \;\vee\; 3 \le x \le 6 \\ 2 \le y \le x+1 \;\vee\; 2 \le y \le -\frac{2}{3}x + 6 \end{cases}$$

lub

$$D: \begin{cases} y-1 \le x \le -\frac{3}{2}y + 9 \\ 2 \le y \le 4 \end{cases}$$

prosta BC:

$y = ax + b \qquad (3,4) \quad (6,2)$

$$\begin{cases} 4 = 3a + b \\ 2 = 6a + b \end{cases} \qquad 2 = -3a \Rightarrow a = -\frac{2}{3}$$

$b = 6$

$4 = -2 + b$

$y = -\frac{2}{3}x + 6 \Rightarrow$

$$\int_2^4 \left(\int_{y-1}^{-\frac{3}{2}y+9} (x+1)\,dx \right) dy = \ldots$$

$$\int\limits_{2}\left(\int\limits_{y-1}(x+1)\,dx\right)dy = \ldots$$

$$\underbrace{y = -\frac{2}{3}x + 6} \Rightarrow$$

$4 = -2 + b$

$y - 6 = -\frac{2}{3}x$