Matematyka 2

1. Całkowanie funkcji trygonometrycznych

1. Całki funkcji trygonometrycznych możemy sprowadzić do całek funkcji wymiernych przy pomocy podstawienia uniwersalnego (tangens połówkowy):

$$t = \operatorname{tg}\frac{x}{2} \Rightarrow \frac{x}{2} = \operatorname{arctg} t \quad \Big|\cdot 2$$

$$x = 2\operatorname{arctg} t$$

$$dx = \frac{2}{1+t^2}\,dt$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}.$$

$$\frac{2\cdot\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cancel{\cos\frac{x}{2}}}\cdot\cos^{\cancel{2}}\frac{x}{2}}{\underbrace{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}_{1}} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$$

2. Jeśli funkcje sin x i cos x występują w parzystych potęgach, stosujemy podstawienie tangensowe:

$$t = \operatorname{tg} x$$

$$dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{t^2}{1+t^2}, \quad \cos^2 x = \frac{1}{1+t^2}.$$

$$\sin x\cdot\cos x = \frac{t}{1+t^2}$$

Przykłady

1. $$\int\frac{dx}{2+\cos x} = \left\{\begin{array}{l} t=\operatorname{tg}\frac{x}{2} \\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2}{1+t^2}\,dt \end{array}\right\} = \int\frac{\frac{2}{1+t^2}\,dt}{2+\frac{1-t^2}{1+t^2}} = \int\frac{2}{\cancel{1+t^2}}\cdot\frac{\cancel{1+t^2}}{2+2t^2+1-t^2}\,dt =$$

$$= 2\int\frac{dt}{t^2+3} = \boxed{\text{wzór } \int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a}+C \quad \text{dla } a=\sqrt{3}} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}}+C =$$

$$= \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\operatorname{tg}\frac{x}{2}\right)+C, \quad C\in\mathbb{R}$$

2. $\int \frac{\sin x \cos x \, dx}{1+\sin^4 x} = \left| \begin{array}{l} t = \operatorname{tg} x \\ dx = \frac{1}{1+t^2}\,dt \\ \sin^2 x = \frac{t^2}{1+t^2} \Rightarrow \sin^4 x = \frac{t^4}{(1+t^2)^2} \\ \sin x \cos x = \frac{t}{1+t^2} \end{array} \right| = \int \frac{\frac{t}{1+t^2}}{1+\frac{t^4}{(1+t^2)^2}} \frac{1}{1+t^2}\,dt =$

$= \int \frac{\frac{t}{(1+t^2)^2}\,dt}{\frac{(1+t^2)^2+t^4}{(1+t^2)^2}} = \int \frac{t}{1+2t^2+t^4+t^4}\,dt = \int \frac{t}{2t^4+2t^2+1}\,dt = \left| \begin{array}{l} t^2 = u \\ 2t\,dt = du \\ t\,dt = \frac{1}{2}du \end{array} \right|$

$= \int \frac{\frac{1}{2}du}{2u^2+2u+1} = \frac{1}{4}\int \frac{du}{u^2+u+\frac{1}{2}} = \frac{1}{4}\int \frac{1}{(u+\frac{1}{2})^2+\frac{1}{4}}\,du =$

$u^2+u+\frac{1}{4}$

$= \left| \begin{array}{l} u+\frac{1}{2} = v \\ du = dv \end{array} \right| = \frac{1}{4}\int \frac{1}{v^2+(\frac{1}{2})^2}\,dv = \frac{1}{4}\cdot\frac{1}{\frac{1}{2}} \operatorname{arctg}\frac{v}{\frac{1}{2}} + C = \frac{1}{2}\operatorname{arctg} 2v + C$

wzór 15, $a = \frac{1}{2}$

$= \frac{1}{2}\operatorname{arctg}(2u+1) + C = \frac{1}{2}\operatorname{arctg}(2t^2+1) + C = \frac{1}{2}\operatorname{arctg}(2\operatorname{tg}^2 x + 1) + C, \quad C \in \mathbb{R}$

Całki z ważniejszych funkcji trygonometrycznych

Wzór	Zakres zmienności
$\int \operatorname{tg} x\,dx = -\ln\lvert\cos x\rvert + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \boldsymbol{Z}$
$\int \operatorname{ctg} x\,dx = \ln\lvert\sin x\rvert + C$	$x \neq k\pi, k \in \boldsymbol{Z}$
$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$	$x \in \boldsymbol{R}$
$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$	$x \in \boldsymbol{R}$
$\int \sin^3 x\,dx = -\cos x + \frac{\cos^3 x}{3} + C$ (potęga nieparzysta)	$x \in \boldsymbol{R}$
$\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3} + C$	$x \in \boldsymbol{R}$
$\int \frac{dx}{\sin x} = \ln\left\lvert\operatorname{tg}\frac{x}{2}\right\rvert + C$	$x \neq k\pi, k \in \boldsymbol{Z}$
$\int \frac{dx}{\cos x} = \ln\left\lvert\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\rvert + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \boldsymbol{Z}$
$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left\lvert\operatorname{tg}\frac{x}{2}\right\rvert + C$	$x \neq k\pi, k \in \boldsymbol{Z}$
$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left\lvert\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\rvert + C$	$x \neq \frac{\pi}{2} + k\pi, k \in \boldsymbol{Z}$

$\int \frac{f'(x)}{f(x)} = \ln\lvert f(x)\rvert$

$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln\lvert\cos x\rvert + C$

$1-\cos^2 x$

$\int \sin^2 x \sin x\,dx = \left\{\begin{array}{l}\cos x = t \\ \sin x\,dx = -dt\end{array}\right.$

Wzory rekurencyjne:

$$\begin{aligned}\int \sin^n x\,dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \quad n \geq 2,\\ \int \cos^n x\,dx &= \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx, \quad n \geq 2,\end{aligned}$$

stosujemy dla n parzystych

Całkowanie funkcji postaci sin ax cos bx, sin ax sin bx, cos ax cos bx.

Do obliczania powyższych całek stosujemy wzory trygonometryczne:

$$1^\circ \quad \sin ax \cos bx = \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right],$$

$$2^\circ \quad \sin ax \sin bx = \frac{1}{2}\left[\cos(a-b)x - \cos(a+b)x\right],$$

$$\cos ax \cos bx = \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right].$$

Przykłady

$\sin(-\alpha) = -\sin\alpha \rightarrow$ nieparzystość

1° $\int \sin \overset{a=2}{2x} \cos \overset{b=4}{4x}\, dx = \frac{1}{2}\int\left[\sin 6x + \sin(-2)x\right]dx = \frac{1}{2}\int\left[\sin 6x - \sin 2x\right]dx =$

$= \frac{1}{2}\int \sin 6x\,dx - \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x + C, \quad C \in \mathbb{R}$

$$\int \sin kx\, dx = -\frac{1}{k}\cos kx\, dx$$

2° $\int \sin \overset{a=1}{x} \sin \overset{b=3}{3x}\, dx = \int \frac{1}{2}\left[\cos(-2)x - \cos 4x\right]dx = \frac{1}{2}\int \cos 2x\,dx - \frac{1}{2}\int \cos 4x\,dx =$

$\cos(-\alpha x) = \cos\alpha x$ parzystość

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C$$

$= \frac{1}{2}\cdot\frac{1}{2}\sin 2x - \frac{1}{2}\cdot\frac{1}{4}\sin 4x + C = \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

Całkowanie funkcji z niewymiernościami

Przydatne wzory:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + c \rightarrow (12)$$

$$\int \frac{1}{\sqrt{x^2+k}}\,dx = \ln\left|x + \sqrt{x^2+k}\right| + c \rightarrow (13)$$

$k \in \mathbb{R}$

1) $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \left\{\begin{array}{l} x = t^6 \\ dx = 6t^5\,dt \\ t = \sqrt[6]{x} \end{array}\right\} = \int \frac{6t^5\,dt}{(t^6)^{\frac{1}{2}} + (t^6)^{\frac{1}{3}}} = 6\int \frac{t^5}{t^3+t^2}\,dt =$

$\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{\frac{1}{3}}$

$= 6\int \frac{t^5}{t^2(t+1)}\,dt = 6\underbrace{\int \frac{t^3}{t+1}\,dt}_{} = (*)$

całka wymierna niewłaściwa

Dzielimy $t^3 : (t+1) = t^2 - t + 1$

$$\begin{array}{l} -t^3 - t^2 \\ \hline = -t^2 \\ \quad t^2 + t \\ \hline \quad = t \\ \qquad -t - 1 \\ \hline \qquad = -1 \end{array}$$

$$(*) = 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt = 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + C =$$

$$= 2\sqrt[6]{x^3} - 3\sqrt[6]{x^2} + \sqrt[6]{x} - \ln|\sqrt[6]{x} + 1| + C$$

2) $\int \sqrt[3]{\frac{x+1}{x-1}}\frac{dx}{x+1} = \left\{ \sqrt[3]{\frac{x+1}{x-1}} = t \; \Big|^3 \right.$

$$\frac{x+1}{x-1} = t^3$$

$$x+1 = t^3(x-1)$$

$$x - t^3x = -t^3 - 1$$

$$x(1-t^3) = -(t^3+1) \,|\, : (1-t^3)$$

$$x = \frac{-(t^3+1)}{1-t^3} \Leftrightarrow x = \frac{t^3+1}{t^3-1} \Leftrightarrow x = \frac{t^3-1+2}{t^3-1}$$

$$x = 1 + \frac{2}{t^3-1}$$

$$dx = \frac{-6t^2}{(t^3-1)^2}dt \qquad dx = \frac{-2\cdot 3t^2}{(t^3-1)^2}dt \Big\} =$$

$$= \int t \frac{\frac{-6t^2}{(t^3-1)^2}dt}{2 + \frac{2}{t^3-1}} = -6\int \frac{\frac{t^3}{(t^3-1)^2}dt}{\frac{2t^3}{t^3-1}} = -6\int\frac{\cancel{t^3}}{(t^3-1)^2}\cdot\frac{t^3-1}{2\cancel{t^3}}dt =$$

$$= -3\underbrace{\int\frac{1}{t^3-1}dt}_{\mathcal{Y}} \qquad \frac{1}{t^3-1} = \frac{1}{(t-1)(t^2+t+1)}$$

$$\mathcal{Y}: \quad \frac{1}{t^3-1} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \quad /\cdot t^3-1$$

$$1 = A(t^2+t+1) + (Bt+C)(t-1)$$

$$1 = At^2 + At + A + Bt^2 - Bt + Ct - C \Rightarrow \begin{array}{l} t^2: \\ t: \\ W: \end{array}\left\{\begin{array}{l} 0 = A + B \\ 0 = A - B + C \\ 1 = A - C \Rightarrow C = A - 1 \end{array}\right.$$

$$+\left\{\begin{array}{l} 0 = 2A + C \\ 1 = A - C \end{array}\right. \Rightarrow 1 = 3A$$

$$A = \frac{1}{3} \qquad B = -\frac{1}{3} \qquad C = -\frac{2}{3}$$

$$\mathcal{Y} = -3\left[\int\left(\frac{\frac{1}{3}}{t-1} + \frac{-\frac{1}{3}t - \frac{2}{3}}{t^2+t+1}\right)dt\right] = -3\left[\frac{1}{3}\int\frac{1}{t-1}dt - \frac{1}{3}\int\frac{t+2}{t^2+t+1}dt\right] =$$

$$= -\int\frac{1}{t-1}dt + \underbrace{\int\frac{t+2}{t^2+t+1}dt}_{\mathcal{Y}_1} = -\ln|t-1| + \mathcal{Y}_1 = -\ln\left|\sqrt[3]{\frac{x+1}{x-1}}\right| + \mathcal{Y}_1$$

$$\mathcal{Y}_1 = \int\frac{\frac{1}{2}(2t+1) + \frac{3}{2}}{t^2+t+1}dt = \frac{1}{2}\int\frac{2t+1}{t^2+t+1}dt + \frac{3}{2}\int\frac{1}{t^2+t+1}dt =$$

$$= \frac{1}{2}\ln|t^2+t+1| + \frac{3}{2}\int \underbrace{\frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}}_{15^\circ} dt = \frac{1}{2}\ln|t^2+t+1| + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2(t+\frac{1}{2})}{\sqrt{3}} + C =$$

$$a = \frac{\sqrt{3}}{2} \quad \Big| = \frac{1}{2}\ln\left|\left(\sqrt[3]{\frac{x+1}{x-1}}\right)^2 + \sqrt[3]{\frac{x+1}{x-1}} + 1\right| + \sqrt{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}}$$

gdzie $t = \sqrt[3]{\frac{x+1}{x-1}}$

3)

$$\int \frac{x+2}{\sqrt{x^2+2x+4}}dx =$$

$$= \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+4}}dx = \frac{1}{2}\underbrace{\int\frac{2x+2}{\sqrt{x^2+2x+4}}dx} + \underbrace{\int\frac{1}{\sqrt{x^2+2x+4}}dx}_{y} = \sqrt{x^2+2x+4} + y$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$$

$$y = \int \frac{1}{\sqrt{(x+1)^2+3}}dx = \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x+1}{\sqrt{3}} + C$$

wzór 15

$a = \sqrt{3}$

4)

$$\int \frac{1-x}{\sqrt{4+2x-x^2}}dx = \int \frac{\frac{1}{2}(2-2x)}{\sqrt{4+2x-x^2}}dx = \frac{1}{2}\int\frac{2-2x}{\sqrt{4+2x-x^2}}dx = \frac{1}{2}\cdot 2\sqrt{4+2x-x^2} + C =$$

$$= \sqrt{4+2x-x^2} + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C$$

5)

$$\int \frac{dx}{\sqrt{4+2x-x^2}} = \int\frac{dx}{\sqrt{5-(x-1)^2}} = \left\{\begin{matrix} x-1=t \\ dx=dt\end{matrix}\right\} = \int\frac{dt}{\sqrt{(\sqrt{5})^2-t^2}} \xrightarrow{\text{wzór 12}}$$

$$= \arcsin\frac{t}{\sqrt{5}} + C = \arcsin\frac{x-1}{\sqrt{5}} + C$$

Twierdzenie 3.1 (metoda współczynników nieoznaczonych)

Do obliczenia całki postaci:

$$\int \frac{W_n(x)}{\sqrt{ax^2+bx+c}}dx,$$

stosujemy wzór:

$$\int \frac{W_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \int\frac{A}{\sqrt{ax^2+bx+c}}dx, \text{ gdzie } a \neq 0.$$

Q_{n-1} – nieznane, A – stała

$W_n(x)$ – dany wielomian stopnia n;

$Q_{n-1}(x)$ – szukany wielomian stopnia n-1 o nieznanych współczynnikach;

A – szukana liczba rzeczywista.

$$\int \frac{x^2+4x+1}{\sqrt{x^2+2x+6}}dx = (mx+n)\cdot\sqrt{x^2+2x+6} + \int \frac{A}{\sqrt{x^2+2x+6}}dx \quad \Big|$$

różniczkujemy obie strony równania

$W_n = x^2+4x+1$

$Q_{n-1} = mx+n, \quad m,n \in \mathbb{R}$

$$\frac{x^2+4x+1}{\sqrt{x^2+2x+6}} = m\cdot\sqrt{x^2+2x+6} + (mx+n)\frac{2x+2}{2\sqrt{x^2+2x+6}} + \frac{A}{\sqrt{x^2+2x+6}} \quad \Big| \cdot\sqrt{x^2+2x+6}$$

$$x^2+4x+1 = m(x^2+2x+6) + (mx+n)(x+1) + A$$

$$x^2+4x+1 = mx^2 + 2mx + 6m + mx^2 + mx + nx + n + A$$

$x^2: \quad 1 = 2m \Rightarrow m = \frac{1}{2}$

$x: \quad 4 = 3m + n \Rightarrow n = 4 - \frac{3}{2}, \quad n = \frac{5}{2}$

$w: \quad 1 = 6m + n + A \Rightarrow A = 1 - 3 - \frac{5}{2} \Rightarrow A = -\frac{9}{2}$

$$\int \frac{x^2+4x+1}{\sqrt{x^2+2x+6}}dx = \left(\frac{1}{2}x+\frac{5}{2}\right)\sqrt{x^2+2x+6} - \frac{9}{2}\underbrace{\int\frac{1}{\sqrt{x^2+2x+6}}dx}_{y}$$

$$y = \int \frac{1}{\sqrt{(x+1)^2+5}} \overset{(13)}{=} \ln\left|x+1+\sqrt{x^2+2x+6}\right| + C$$

$$(13) \quad \int \frac{1}{\sqrt{x^2+k}}dx = \ln\left|x+\sqrt{x^2+k}\right| + C$$

4. Podstawienia Eulera

Twierdzenie 4.1

Całkę z funkcji zawierającej x oraz $\sqrt{ax^2+bx+c}$,

gdzie a ≠0 i $\Delta = b^2 - 4ac$ ≠0, sprowadzamy do całki z funkcji wymiernej stosując podstawienia:

I. (pierwsze podstawienie Eulera)

$$\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}, \quad \text{jeśli } a > 0;$$

II. (drugie podstawienie Eulera)

$$\sqrt{ax^2+bx+c} = xt - \sqrt{c}, \quad \text{jeśli } c > 0;$$

III. (trzecie podstawienie Eulera)

$$\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = x(x-x_1)t, \quad \text{jeśli } \Delta > 0.$$

W podstawieniu II, różnicę $xt - \sqrt{c}$ można zastąpić sumą $xt + \sqrt{c}$, lub różnicą $\sqrt{c} - xt$.

Stosując I podstawienie Eulera obliczymy całkę $\left(a = 1 \Rightarrow \sqrt{} = t - x \right.$

$$\int \frac{dx}{x\sqrt{x^2-2x}} = \left\{ \begin{array}{l} \sqrt{x^2-2x} = t - x \quad |()^2 \Rightarrow t = x + \sqrt{x^2-2x} \\ \cancel{x^2} - 2x = t^2 - 2tx + \cancel{x^2} \\ -2x + 2tx = t^2 \\ x(2t-2) = t^2 \Rightarrow x = \dfrac{t^2}{2(t-1)} \\ dx = \dfrac{2t \cdot 2(t-1) - t^2 \cdot 2}{4(t-1)^2}\,dt \\ dx = \dfrac{4t^2 - 4t - 2t^2}{4(t-1)^2}\,dt \\ dx = \dfrac{2(t^2-2t)}{4(t-1)^2}\,dt \\ dx = \dfrac{t^2-2t}{2(t-1)^2}\,dt \end{array} \right\} =$$

$$= \int \frac{\frac{t^2-2t}{2(t-1)^2}\,dt}{\frac{t^2}{2(t-1)} \cdot \left(t - \frac{t^2}{2(t-1)}\right)} = \int \frac{\frac{t^2-2t}{2(t-1)^2}\,dt}{\frac{t^3}{2(t-1)} - \frac{t^4}{4(t-1)^2}} = \int \frac{t^2-2t}{\cancel{2}\cancel{(t-1)^2}} \cdot \frac{\overset{2}{\cancel{4}}\cancel{(t-1)^2}}{t^3 \cdot 2(t-1) - t^4}\,dt =$$

$$= \int \frac{t^2-2t}{t^4 - t^3 - t^4}\,dt = \int \frac{t^2-2t}{-t^3}\,dt = -\int \frac{1}{t}\,dt + 2\int \frac{1}{t^2}\,dt =$$

$$= -\ln|t| - \frac{2}{t} + C = -\ln\left|x + \sqrt{x^2-2x}\right| - \frac{2}{x + \sqrt{x^2-2x}} + C$$

$$\nabla\; y = \int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int x \cdot \frac{x}{\sqrt{x^2+1}}\,dx = \left| \begin{array}{ll} f'(x) = \frac{2x}{\sqrt{x^2+1}} \cdot \frac{1}{2}, & g(x) = x \\ \Downarrow & \Downarrow \\ f(x) = \sqrt{x^2+1} & g'(x) = 1 \end{array} \right| = \left(\int \frac{f'(x)}{\sqrt{f(x)}} = 2\sqrt{f(x)} \right)$$

$$= x\sqrt{x^2+1} - \int \sqrt{x^2+1} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\,dx = x\sqrt{x^2+1} - \int \frac{x^2+1}{\sqrt{x^2+1}}\,dx$$

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx + \int \frac{1}{\sqrt{1+x^2}}\,dx$$

$$y = x\sqrt{x^2+1} - \underbrace{\int \frac{x^2}{\sqrt{x^2+1}}\,dx}_{y} - \int \frac{1}{\sqrt{x^2+1}}\,dx$$

$$2y = x\sqrt{x^2+1} - \int \frac{1}{\sqrt{x^2+1}}\,dx \quad 13 \quad \Big| \cdot \frac{1}{2}$$

$$y = \frac{1}{2} x\sqrt{x^2+1} - \frac{1}{2}\ln\left|x + \sqrt{x^2+1}\right| + C$$

$$\nabla\; \int \frac{1}{x\sqrt{x^2+1}}\,dx = \int \frac{1}{x\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}\,dx = \int \frac{1}{x^2 \cdot \sqrt{1+\frac{1}{x^2}}}\,dx = \left| \begin{array}{l} \frac{1}{x} = t \\ -\frac{1}{x^2}\,dx = dt \\ \frac{1}{x^2}\,dx = -dt \end{array} \right|$$

$$= \int \frac{-dt}{\sqrt{1+t^2}} = -\ln\left|t + \sqrt{1+t^2}\right| + C =$$

$$= -\ln\left|\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right| + C, \quad C \in \mathbb{R}$$